The Standard Model from a Symmetry Perspective

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in the

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Declaration of Authorship

I, Samuel Kay, declare that this project report and the work presented in it are my own. I confirm that:

- I have complied with the Department of Mathematical Sciences' guidance on multiple submission and on the use of AI tools;
- Where I have consulted material from the work of others not involved in the project, this is always clearly attributed;
- Where I have quoted and paraphrased from the work of others, the source is always given;
- I have acknowledged all main sources of help;
- All uses of AI tools have been declared.

Signed:

Date: 2nd May 2025

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Going back a year, I thank Dr Andreas Braun for his excellent and undoubtedly enthusiastic presentation of the third year module 'Geometry of Mathematical Physics III' here at Durham. This module single-handedly inspired me to read more about the mathematics of particle physics and how, really, all we know about the universe is that it presents us with nice symmetries. I originally only took the module because I wanted to study Advanced Quantum Theory IV in the following year, and GMP was a prerequisite, but one can only imagine how much enjoyment and knowledge I might have missed out on if that had not been the case.

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Finally, my mental wellbeing would not be as healthy if it were not for the love and support from all four of my parents, Mum, Dad, Anna & Chris, and Daisy, my rock through this entire process.

FIGURE 1: A group photo of the 2024/25 MMath cohort at Durham University. Notable SW3 residents on the front row include persons one and three, Oli and Louie, myself in fourth place, and Tom and Tom as persons five and six.

Contents

Declaration of Authorship i

Acknowledgements ii

1 Introduction 1
- 1.1 The search for matter and forces 1
- 1.2 Constructing the Standard Model 2

2 Group Structures, Symmetries and Representations 4
- 2.1 The Algebra of Physical Space 4
  - 2.1.1 Pauli vectors 6
  - 2.1.2 Higher-dimensional algebras 7
- 2.2 The Spacetime Algebra 8
- 2.3 Symmetries of spacetime 8
  - 2.3.1 The Poincaré algebra 10
  - 2.3.2 Conserved quantities 13
- 2.4 Particles as representations 14
  - 2.4.1 Irreducible representations of the Lorentz group 14
  - 2.4.2 Unitary irreducible representations of the Poincaré group 15
  - 2.4.3 Wigner's classification 17
- 2.5 Spin-1/2 representations of the Lorentz group 19
  - 2.5.1 Chirality 22
  - 2.5.2 Dirac spinors 23

3 The Standard Model is a Field Theory 25
- 3.1 Quantum field theory 25
  - 3.1.1 Why field theory? 26
  - 3.1.2 Poincaré transformations on fields 26
  - 3.1.3 The Lagrangian formalism 28
- 3.2 Internal symmetries 29

4 Spontaneous Symmetry Breaking 31
- 4.1 Global breaking of symmetries 31
  - 4.1.1 Discrete symmetry breaking 31
  - 4.1.2 Continuous symmetry breaking 33
  - 4.1.3 Faster than lightspeed—not! 34
- 4.2 Fields in higher dimensions 35
  - 4.2.1 O(N) sigma models 35
  - 4.2.2 Goldstone's theorem 37
- 4.3 The Higgs mechanism 38
  - 4.3.1 Gauge invariance 39
  - 4.3.2 Gauge symmetry breaking and the Abelian Higgs model 40
  - 4.3.3 The photon mass: can we fix it? 42

5 The Weak Force as a Gauge Theory **43**
5.1 Yang-Mills fields 43
5.2 Electroweak symmetry breaking 45
5.2.1 Applying the SU(2) algebra 46
5.2.2 The Glashow-Weinberg-Salam model: a first look 46
5.2.3 The photon mass: yes we can! 49
5.2.4 Electric charge 50
5.3 Electroweak interactions with fermions 52
5.3.1 Quarks and leptons 52
5.3.2 The Glashow-Weinberg-Salam model: a second look 53

6 Conclusion **54**
6.1 Overview of the report 54
6.2 How does the Standard Model progress? 55

References **56**
Articles 56
Books 57
Lecture notes and theses 58
Journal articles 59
Videos 62
Web pages 62

A The Standard Model Lagrangian **64**

B Simplifying the Kinetic Higgs Term in Glashow-Weinberg-Salam Theory **66**

1 Introduction

The Standard Model of Particle Physics is often described as the fundamental theory of Nature [45, p. 3]. It is, to date, the most successful description of particle physics the modern world has to offer. In a physical sense it is a theory that describes all known fundamental particles and their interactions in terms of relativistic quantum field theories, that can be (and have been) verified by experiment. This fact alone makes the Standard Model seem extremely complicated—and that would be a correct judgement; the Lagrangian for this theory contains at least 74 terms when expanded in the way as in Appendix A. It is the culmination of decades of dedicated observation, theory and experiment [45, p. 3].

From a mathematical point of view, one can understand the beginnings of the Standard Model as being reliant on the idea that the spacetime we live in obeys certain symmetries. The core aim of this report is to motivate the construction of the Standard Model from the first principles of symmetry, which we know to be the underlying proponent of group theory. Before we jump into any rigour, though, we will get an insight into why such measures were taken to construct such a theory.

1.1 The search for matter and forces

Humankind has always been fascinated by what *stuff* is made up of and the question dates back millennia. In our history there have been many schools of thought as how to describe matter with intuition [90]. One initially successful idea led by Leucippus and Democritus proposed that matter is discrete, is made up of indivisible parts, named *atoms* [24, p. 31-33]. These were later, more aptly named *particles*, which would be held together by and interact via various forces. This theory carried us through classical physics and allowed for the discoveries of various particles, famoulsy the electron by Thompson, the proton by Rutherford and the neutron by Chadwick, etc [4, 76, 81].

Another successful approach didn't come until many centuries after, this being that matter could be continuously distributed; that there exist *fields* permeating space that carry information such as mass, momentum, energy and spin. It is hard to pinpoint the origins of this idea exactly since many physicists in the early 20$^{\underline{\text{th}}}$ century were working on this idea, but one notable person to highlight would be Louis de Broglie. In 1924, he proposed that electrons (and therefore all 'particles' of matter) experienced wave-like behaviour [35]. This paved the way for the earliest theories of quantum mechanics, with Schrödinger in 1926 giving us his equation for how the probability amplitude of a particle's wavefunction changes over time [79, p. 1068], and then with Dirac in 1928 who aimed for a relativistic wave equation that was specific to fermions like the electron [60, p. 615]. It was then thought these wavefunctions propagate through fields, which began the study of quantum field theory.

One other note is that this description of wave-particle duality has its origins in light. Newton had claimed light showed frequency-like properties and therefore would appear in discrete amounts, but this was contradictory to what Huygens and Young believed in a similar time frame, which was that the effects of light mimic that of pebbles thrown in water; ripple-like patterns emerge and their interference increases light intensity. Further experiments (such as ones involving the photoelectric effect and double-slit experiment) verified that light exhibits both properties. Although light may naturally exist as waves, its energy is quantised in packets, as particles, called 'photons'. In was then proposed in theories developed by Faraday and Maxwell that light itself carries the electromagnetic force, and that interactions between charged particles are due to the exchange of photons [13, p. 227-243]. In this sense we have that the forces between particles are themselves particles, which could alternatively be described via fields.

As it turns out, both the particle and field descriptions of matter and forces are correct. But also neither description is correct. In Leonard Susskind's lecture series on the Standard Model [90] he says "there are subtleties in the study of quantum physics that satisfy both ideas, but leave some bits of either one unsatisfied". It has therefore been desired for a long time for there to exist a natural connection between particles, fields and forces that unify all of the theories into one, beautiful **theory of everything**.

1.2 Constructing the Standard Model

What physicists and mathematicians were fixated on in the 1970s was a way to categorise all known fundamental particles and their interactions (ignoring gravity) based on both theory and experimentation [3]. This was propelled by the development of electroweak theory, whereby in standard science fashion the theory [64, 78, 82] was confirmed by experiment [2]. It verified that what we once thought about electromagentic and weak interactions were, in fact, two separate manifestations of the very same theory.

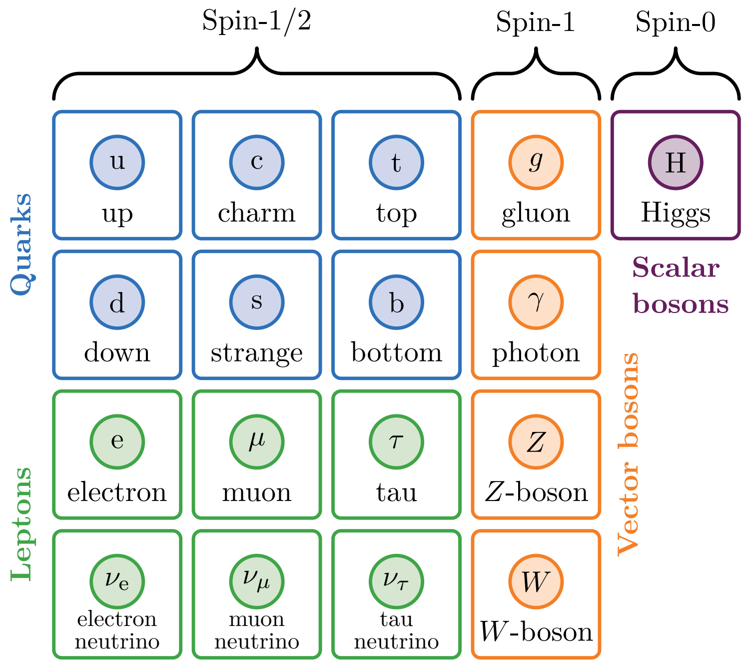

FIGURE 1.1: A simplified diagram of particles in the Standard Model. Adapted from [7].

In this report we will examine the theory that led 20$^{\text{th}}$ century scientists towards this fact, with a strong focus on the **symmetries** behind the Standard Model that we will uncover along the way. The reason for this is that mathematical symmetries are intimately linked with conservation laws, as famously postulated by Noether in 1918 [75, p. 238-239]. We begin in Chapter 2 by understanding why we are allowed to use mathematics in the context of particle physics. In particular, we use techniques from representation theory to see that the concept of a particle arises naturally in a group-theoretic manner and that different representations of the symmetry group of spacetime allow us to classify them by their spins. These different representations predict particles of spin-1 (the vector bosons), spin-1/2 (the fermions) and spin-0 (the scalar bosons), as summarised in Figure 1.1. The 'symmetry group of spacetime' refers to the group of coordinate transformations that preserves flat spacetime;

the scales at which we will be working with allow for the curvature of spacetime to be ignored. These are a combined effort of translations, rotations and Lorentz boosts in spacetime.

This continues in Chapter 3 where we justify moving from a particle framework to a field framework. The notion of the Standard Model being a theory of fields will be shown to be a *necessity* of the fact that these fundamental particles must obey the laws of quantum mechanics *and* special relativity. This introduces us to another class of symmetries that act internally on fields to relate them to one other, or itself, but do not involve spacetime. These internal symmetry groups will become the mathematical way to describe how fields interact with each other. More specifically, the maximal internal symmetry group G for the Standard Model is

$$\mathrm{G} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3), \tag{1.2.1}$$

which we interpret to mediate the three fundamental forces of Nature: the electromagnetic force, weak nuclear force and strong nuclear force. All of the above will allow us to categorise all fundamental particles by their theoretical masses, spins and which internal symmetry groups they interact with, which can then be used to contruct a model almost like the Periodic Table.

► The reason we say 'maximal' is that we technically don't know the group but we do know its algebra. Other allowed groups for consideration are $\mathrm{G}/\mathbb{Z}_N$ for $N = 1, 2, 3, 6$ [49, p. 193-194].

An issue that we will see arise in Chapter 2 is that the theory we construct is for massless particles only, when in reality we know there exist many fundamental particles that carry a mass. The way in which we hope to fix this issue in discussed in Chapter 4 and relies on the concept of symmetry *breaking*, whereby there may exist a specific term in the Lagrangian of a field theory that will violate the symmetry it was constructed with when considering its ground state. Through this, we uncover the Higgs boson: a spin-0 particle that reamined a theoretical mystery until its discovery in 2012 [58], 48 years after its first proposal [55, 68, 69].

In Chapter 5 we will understand how the electroweak theory is formulated. This comes in the form of first identifying the correct symmetry group and attempting to write down a Lagrangian that stays consistant with symmetries of spacetime, to then use ideas of symmetry breaking that separates the massive gauge bosons that govern the weak force and the massless photon that mediates the electromagnetic force. Once through with that, we discuss its interactions with relevant fermions in the Standard Model and conclude that we have all of the theoretical tools necessary to build the theory of everything, excluding gravity.

Inspired conventions

In this report we use the metric convention

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1), \tag{1.2.2}$$

which differs from a lot of sources regarding special relativity and quantum field theory, and hence about the Standard Model. As opposed to [15, p. x], it may have been more useful (and perhaps easier) to write this report with the more standardised notation, but the habits of undergraduate study are not so easily broken.

Wherever a result or framework has been introduced by a notable figure, a footnote is given with more information about said figure, inspired by Osborn[a] in [43].

[a] Hugh Osborn, British.

2
Group Structures, Symmetries and Representations

The goal of this chapter is to use group theory to uncover the symmetries that allow us to describe elementary particles and their interactions. These come in the form of symmetries of spacetime itself and **internal symmetries**, though this chapter will focus only on the former—see Chapter 3 for internal symmetries.

We begin in Section 2.1 with understanding the foundation on which the Standard Model is built on: geometric algebras in many dimensions. These are abstract algebras that almost seem like the 'mother' of all other algebras—a lot of the time, an algebra we know and love is just a sub-algebra of an existing geometric algebra. This is extended in Section 2.2 to an algebra that allows us to describe spacetime. We then move to the algebra and structure of *transformations* in spacetime in Section 2.3. Without this, there would be no understanding of how particles interact with each other. Once the algebra is determined we can use symmetry arguments and find out what the symmetries of spacetime mean physically in terms of their conservation laws.

Our newfound understanding of spacetime symmetries will lead us down a pathway of believing that group structures and particle interactions are one and the same, but we shouldn't automatically believe that is the case! In Section 2.4 we provide formal reasoning as to why it will be fruitful to describe particles and their interactions with algebraic structures. In fact, we will be able to describe particles as **irreducible unitary representations** of a specific group of spacetime transformations. With this in mind we discuss in detail spin-1/2 representations of the Lorentz group in Section 2.5 that will allows us to transform spinor-valued objects.

2.1 The Algebra of Physical Space

A lot of time was spent in previous courses understanding Lie groups and Lie algebras, with a heavy focus on SU(2) because it has lots of natural applications outside of weak interactions, which is where we shall spend a lot of our time (see Chapter 5). One of these makes itself very apparent in quantum mechanics whereby angular momentum plays a key role in understanding what 'spin' is; the $\underline{\mathbf{3}}$ of SU(2) (more commonly seen in the literature as SO(3)) acts on particles to rotate them through 3D space, whereas the $\underline{\mathbf{2}}$ of SU(2) acts to internally rotate quantum spin states. This is well-known and it is convenient to describe its algebra $\mathfrak{su}(2) = \{\mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3\}$ with a basis of Pauli[1] matrices. These are the most general form of traceless 2×2 hermitian matrices with a neat Lie bracket:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \implies [\sigma_a, \sigma_b] = 2\mathrm{i}\varepsilon_{abc}\sigma_c. \tag{2.1.1}$$

This is not, however, the full story. One can extend the role of the σ's by considering its **Clifford**[2] **algebra**: the algebra constructed from the σ's and their vector products. This doesn't seem like a useful tool at first, but once the dust settles we will see that many different structures familiar to us are manifestations and sub-algebras of one single Clifford algebra. We introduce these algebras, sometimes referred to as **geometric algebras**, and then find a way to include Pauli matrices as part

[1] Wolfgang Pauli, 1900 - 1958, Austrian, Nobel Prize 1945.
[2] William K. Clifford, 1845 - 1879, English.

of this, following Sections 2.1 and 2.3 from McKenzie's thesis on an application of the Algebra of Physical Space [41] and the introduction to Baylis' book [14].

A geometric algebra is an algebra generated from geometric products of orthonormal basis vectors $\{\mathbf{e}_i\}$ in N dimensions, $1 \leqslant i \leqslant N$; elements in this algebra can therefore be linear combinations of the following:

- a *scalar*, as per its usual definition;
- a *vector*, some linear combination of the basis vectors $\{\mathbf{e}_i\}$:

$$\mathbf{v} = v_i \mathbf{e}_i = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \ldots + v_N \mathbf{e}_N; \tag{2.1.2}$$

- a *bivector*, the area element constructed from two distinct vectors with an induced orientation. An area element with basis $\mathbf{e}_i \mathbf{e}_j$ (for $i < j$) is the area element coplanar to vectors of the form $v\mathbf{e}_i + w\mathbf{e}_j$. The orientation is given by the rotation required to align $\mathbf{e}_i$ with $\mathbf{e}_j$;
- a *trivector*, the volume element constructed from three distinct vectors with induced orientation;

$\vdots$

and so forth, with each new product constructing elements of higher spatial dimension. Mathematically speaking, these objects are arrays of varying dimension that map out rotations from one element into another. For example, the bivector $\mathbf{e}_1\mathbf{e}_2$ can be represented by the rotation matrix that sends $\mathbf{e}_1$ to $\mathbf{e}_2$ in that order; that's what gives it its orientation. The number of elements for each product of k basis vectors is $\binom{N}{k}$.

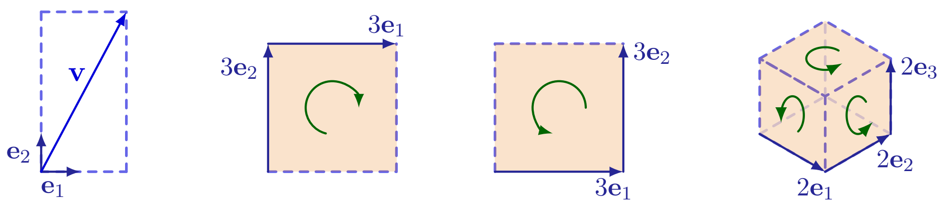

FIGURE 2.1: (L–R) A vector, two bivectors with opposite orientation and a trivector. Inspired by some diagrams from [88].

The geometric product in question is defined so that the product of two vectors can be replaced by the standard dot product to give its squared length,

$$\mathbf{v}\mathbf{v} := \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2. \tag{2.1.3}$$

We can always write a vector as the sum of two other vectors; if $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ then using the above we must have

$$\begin{aligned} (\mathbf{v}_1 + \mathbf{v}_2)(\mathbf{v}_1 + \mathbf{v}_2) &= (\mathbf{v}_1 + \mathbf{v}_2) \cdot (\mathbf{v}_1 + \mathbf{v}_2) \\ \implies \quad \mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2\mathbf{v}_1 &= 2\, \mathbf{v}_1 \cdot \mathbf{v}_2, \end{aligned} \tag{2.1.4}$$

cancelling the common terms using (2.1.3). Since the basis vectors are orthonormal this immediately shows that any two basis vectors anticommute: $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$. This also makes sense in terms of bivectors; the plane segment enclosed by two vectors will reverse its orientation if the vectors are in a different order as in Figure 2.1. This also brings about an *anticommutation* relation satisfied by the basis vectors,

$$\{\mathbf{e}_i, \mathbf{e}_j\} = 2\delta_{ij} \mathbb{1}_N, \tag{2.1.5}$$

evident from the fact that we have orthonormality. It should also be clear that the highest hypervolume element has dimension N. If we consider a possible $(N+1)$-dimensional element, there must be a copy of a basis element somewhere in the basis product. We can always move the two basis elements closer together via repeated anticommutation and cancel them both out with their squared length resulting in an $(N-1)$-dimensional element.

2.1.1 Pauli vectors

Clifford algebras are defined by the number of orthonormal basis vectors they contain and also the sign of what each basis vector squares to. For example, suppose we have one basis element i defined such that $\mathrm{i}^2 = -1$. This creates the Clifford algebra $\mathrm{Cl}_{0,1}$ since there are no basis elements squaring to 1 and one squaring to -1. The elements in $\mathrm{Cl}_{0,1}$ must therefore be constructed from linear combinations of 1 and i; these are nothing but the complex numbers!

With this in mind it makes sense that the algebra required to study real 3D Euclidean space is the Clifford algebra $\mathrm{Cl}_{3,0}(\mathbb{R})$[3] with 3D Cartesian basis vectors as the vector elements, noting $|\mathbf{e}_i|^2 = +1$. This algebra has eight elements (one scalar, three vectors, three bivectors and one trivector) and forms the **Algebra of Physical Space** (APS). Oftentimes the trivector (and indeed the N-dimensional element in a general Clifford algebra) is denoted as a 'pseudoscalar'; it is not scalar by construction but it is not difficult to check that by its very nature it behaves just like the complex i. That is, it squares to -1 and commutes with all other algebra elements.

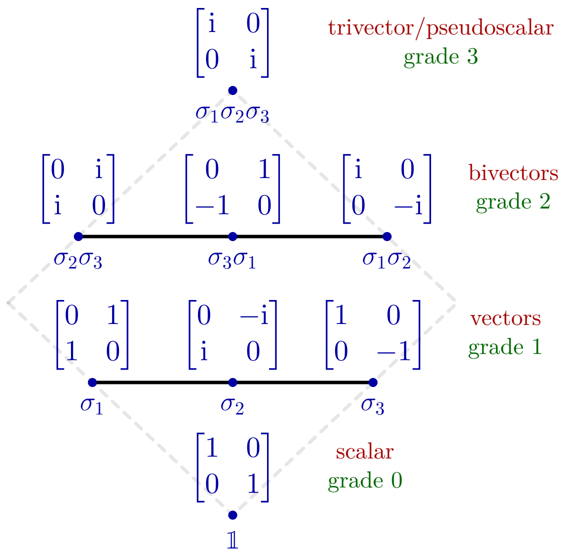

FIGURE 2.2: The Algebra of Physical Space, Cl_3, with Pauli matrices as the basis vectors. Inspired by a diagram from [88].

Using the 3D Cartesian basis vectors is an intuitive starting point but will not be practical for the purposes of this chapter. If we were to use different basis vectors, does there exist a set of three elements already familiar to us that all square to $+1$ and anticommute? The answer is, of course, yes: we use the Pauli matrices as basis vectors. Figure 2.2 presents a graphic of all elements living in Cl_3 with Pauli's basis separated by their *grades*.

This restructuring now seems more obvious given that the geometric product is given by matrix multiplication. It is also even clearer to see why the trivector can be rightfully called a pseudoscalar—it is a rescaling of the identity matrix, so of course it commutes with everything. One might think that this is not a valid representation of APS purely because we have introduced the complex i. This is not to be feared, though; we can always return to 3D Euclidean space by writing any vector $\mathbf{v}$ as a linear combination of the σ's and vice versa. This object we will call a *Pauli vector* X:

$$\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \quad \longleftrightarrow \quad X = x_i\sigma_i = \begin{bmatrix} z & x - \mathrm{i}y \\ x + \mathrm{i}y & -z \end{bmatrix}. \tag{2.1.6}$$

These will make a reprise in Section 2.5 while understanding spin-1/2 representations of certain Lie groups. We can return to the question of why understanding Clifford algebras was a good idea. We

[3]Sometimes the second label is omitted if no elements square to -1, and APS is taken to be real so in fact the $\mathbb{R}$ is often omitted too.

saw that the complex numbers are in fact the Clifford algebra $\mathrm{Cl}_{0,1}$. This can also be generalised to Hamilton's[4] set of quaternions $\mathbb{H} = \{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$, where we identify $\mathbb{H}$ as the even-grade subalgebra of Cl_3. That is, one can map each unit quaternion to the scalar $\mathbb{1}$ and three bivectors $\sigma_i\sigma_j$ since they all have the same properties, namely that all the bivectors square to $-\mathbb{1}$ and

$$\begin{aligned} \mathrm{ijk} &\cong (\sigma_3\sigma_1)(\sigma_2\sigma_3)(\sigma_1\sigma_2) \\ &= (\sigma_1\sigma_2\sigma_3)\sigma_3\sigma_1\sigma_2 \\ &= -\sigma_1\sigma_2\sigma_2\sigma_1 = -\mathbb{1}. \end{aligned} \tag{2.1.7}$$

We will see shortly that the algebra for spacetime transformations is simply the vectors and bivectors from the Pauli representation of Cl_3.

2.1.2 Higher-dimensional algebras

One more useful property of Clifford algebras we shall highlight is the fact that they provide a representation of special orthogonal Lie algebras. We know that for $a = 1, 2, 3$, $\{\mathrm{i}\sigma_a\}$ forms a basis of $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Now, it is clear from Figure 2.2 that $\sigma_b\sigma_c = \mathrm{i}\sigma_a$ for an even permutation of a, b, c. This means that the Lie algebra $\mathfrak{so}(3)$ is formed from the bivectors of Cl_3. This is not a one-time occurance: in fact, the special orthogonal Lie algebra on any vector space V is isomorphic to the set of bivectors in the Clifford algebra of the same vector space, $\mathrm{Cl}(V)$ [99]. The proof of the full statement requires a solid understanding of noncommutative geometry and the reader is directed to a series of lectures on the subject [50]. For now, we take a particular form of the statement and convince ourselves it is true.

Special orthogonal Lie algebra relation to Clifford algebras

For any dimension $N > 1$,

$$\mathfrak{so}(N) \cong \mathrm{Cl}_N^{(2)}, \tag{2.1.8}$$

where we have selected the bivectors, the grade 2 elements, from $\mathrm{Cl}_{N,0}(\mathbb{R})$.

In order to understand this, we first note that the dimensions of each algebra line up: there are $\binom{N}{2} = N(N-1)/2$ in Cl_N, and this must also be the number of elements for $\mathfrak{so}(N)$ because they are the basis of antisymmetric $N \times N$ matrices. We can then choose the basis of each algebra to be the ND Cartesian basis, by which $\mathbf{e}_i\mathbf{e}_j$ will be the rotation matrix that sends $\mathbf{e}_i$ to $\mathbf{e}_j$ in that order. This will be a rotation in the i, j-plane, which we can match up with the basis element $T_{ij} \in \mathfrak{so}(N)$. Then:

$$\mathfrak{so}(N) = \{\mathbf{e}_1\mathbf{e}_2, \ldots, \mathbf{e}_1\mathbf{e}_N, \mathbf{e}_2\mathbf{e}_3, \ldots, \mathbf{e}_2\mathbf{e}_N, \ldots, \mathbf{e}_{N-1}\mathbf{e}_N\}. \tag{2.1.9}$$

If we define E_{ij} to be single-valued matrix whose $i, j^{\underline{\mathrm{th}}}$ entry is 1 and 0 elsewhere, we write

$$(E_{ij})_{kl} := \delta_{ik}\delta_{jl}. \tag{2.1.10}$$

Then any $T_{ij} \in \mathfrak{so}(N)$ may be written as the difference between two of these elements that act in the same plane, namely

$$T_{ij} := -\mathrm{i}(E_{ij} - E_{ji}) \quad \Longrightarrow \quad (T_{ij})_{kl} = -\mathrm{i}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}). \tag{2.1.11}$$

It will be useful to place in this factor of of $-\mathrm{i}$ so that when we exponentiate the generators with some arbitrary parameter $\boldsymbol{\vartheta}$ we end up with a purely real set of matrices $\boldsymbol{L}_{ij} \in \mathrm{SO}(N)$ that are positively oriented:

$$\boldsymbol{L}_{ij} = \mathrm{e}^{\mathrm{i}\boldsymbol{\vartheta}T_{ij}} = \exp\{\vartheta_{kl}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})\}. \tag{2.1.12}$$

[4] William Hamilton, 1805 - 1865, Irish.

These will become useful when studying fields in higher dimensions—see Section 4.2.

2.2 The Spacetime Algebra

In this short section we introduce the Spacetime Algebra (STA) which is a natural extension to APS but for flat spacetime instead of 3D Euclidean space. We know from special relativity that flat spacetime is $\mathbb{R}^4$ equipped with the metric $\eta_{\mu\nu} := \operatorname{diag}(-1, +1, +1, +1)$, often denoted Minkowski[5] spacetime $\mathbb{R}^{1,3}$. The algebra for this space will require four basis vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ that satisfy the anticommutation relation:

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}_4. \tag{2.2.1}$$

From this, we deduce that the vectors anticommute and their squared lengths are

$$(\gamma_0)^2 = -\mathbb{1}_4, \;\; (\gamma_a)^2 = +\mathbb{1}_4 \;\text{ for }\; a = 1, 2, 3. \tag{2.2.2}$$

This gives us the 16-dimensional Clifford algebra $\text{Cl}_{3,1}$. A convenient representation for us to use for the γ's will be the 4×4 Dirac[6] matrices in the **chiral basis** [10]:

$$\gamma_0 = \begin{bmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{bmatrix}, \quad \gamma_a = \begin{bmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{bmatrix}, \tag{2.2.3}$$

for Pauli's matrices σ_a. We will meet other elements of this algebra along the way, but one to make note of now is a renormalised version of the quadvector/pseudoscalar $\gamma_0\gamma_1\gamma_2\gamma_3$ that we call γ_5, defined to be:

$$\gamma_5 := -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = -\mathrm{i}\begin{bmatrix} \sigma_1\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_1\sigma_2\sigma_3 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix}. \tag{2.2.4}$$

The γ's will clear act on objects with four components, but certainly not vectors; one can easily see that the elements of $\text{Cl}_{3,1}$ will always be diagonal or off-diagonal matrices, meaning they form a reducible representation and each 'block' of a matrix will act separately on a two-component object. This is what we realise to be a **spinor**. γ_5 then clearly distinguishes between two 'stacked' spinors by giving one a negative symbol. Finally, a 4-vector can be written with these basis elements as such:

$$x^\mu\gamma_\mu = \begin{bmatrix} 0 & x^0\mathbb{1}_2 + x^a\sigma_a \\ -(x^0\mathbb{1}_2 - x^a\sigma_a) & 0 \end{bmatrix}. \tag{2.2.5}$$

One sees that the way we have written the above shows the bottom-left block of matrices to mimic a *parity* transformation: all spatial elements have been sent to their negative counterparts.

These matrices were introduced by Dirac in 1928 with the goal of writing a relativistic wave equation for spinor fields [60]; the Klein–Gordon[7] equation derived in 1926 is relativistic, but acts on scalar fields [67, 70]. The arguments for spinors will become more precise in Section 2.5 when we discuss in detail spinor representations of the Lorentz[8] group. First, though, we must look at how we can construct such a group from demanding the invariance of spacetime itself.

2.3 Symmetries of spacetime

Spacetime is the field in which all particles exist, should we hope to describe them. Space and time become unified in such a way that allows us to deal with them together while still treating them as fundamentally different dimensions. Particle interactions occur on such small scales that allows us to

[5] Hermann Minkowski, 1864 - 1909, Lithuanian-German.
[6] Paul Dirac, 1902 - 1984, English, Nobel Prize 1933.
[7] Oskar Klein, 1894 - 1977, Swedish & Walter Gordon, 1893 - 1939, German.
[8] Hendrik Lorentz, 1853 - 1928, Dutch, Nobel Prize 1902.

ignore general relativity and only consider flat spacetime. That isn't to say we should ignore gravity; indeed, gravity is a fundamental force and one might suggest it belongs in the Standard Model, but that would require us to understand what particle is associated with gravitational oscillations. The problem we face is that the fundamental forces we can successfully describe via the Standard Model are all manifestations of quantum fields, whereas gravity is the curvature of the spacetime field [92]. The reader is directed to any course on General Relativity to understand this statement more meaningfully [16, 17, 26].

This introduction will follow Section 2.1 of [45] and Chapter 4 of [43]. We begin by studying the symmetry of spacetime in order to unearth its conserved quantities. In any reference frame one defines the invariant interval s^2 to be the metric contracted with the 'norm' of a 4-vector,

$$x^\mu x_\mu = x^\mu \eta_{\mu\nu} x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2, \tag{2.3.1}$$

which is preserved under certain transformations in spacetime, namely Lorentz transformations. The Lorentz group L is the six-dimensional group of linear maps Λ that act on $\mathbb{R}^{1,3}$ to preserve the invariant interval $x^\mu x_\mu$:

$$\begin{aligned} x^\mu \eta_{\mu\nu} x^\nu \;\longmapsto\; & \left(\Lambda^\mu{}_\rho x^\rho\right) \eta_{\mu\nu} \left(\Lambda^\nu{}_\sigma x^\sigma\right) \\ &= x^\rho \left(\Lambda_\rho{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\sigma\right) x^\sigma \\ &\overset{!}{=} x^\rho \eta_{\rho\sigma} x^\sigma . \end{aligned} \tag{2.3.2}$$

This transformation property requires the relation $\Lambda^T \eta \Lambda = \eta$, wherein the second line we could move the first Λ past the 4-vector components by imposing its transpose. With this, we can deduce the remaining properties of the Lorentz group. By taking determinants on both sides we see

$$\det(\Lambda^T \eta \Lambda) = \det(\eta) \implies -(\det \Lambda)^2 = -1 \implies \det \Lambda = \pm 1. \tag{2.3.3}$$

We can go further by separating the time components. If we take the upper-leftmost component in the above relation one observes:

$$(\Lambda^T \eta \Lambda)^0{}_0 = \eta^0{}_0 = -1 \implies -\left(\Lambda^0{}_0\right)^2 + \left(\Lambda^1{}_0\right)^2 + \left(\Lambda^2{}_0\right)^2 + \left(\Lambda^3{}_0\right)^2 = -1. \tag{2.3.4}$$

If the first term has three positive numbers adding to it to obtain -1, then $|\Lambda^0{}_0| \geqslant +1$. This tells us that the Lorentz group can be split into four disjointed components $\{\mathrm{L}_+^\uparrow, \mathrm{L}_-^\uparrow, \mathrm{L}_+^\downarrow, \mathrm{L}_-^\downarrow\}$, each accessible to one another by transformations in the set $\{\mathbb{1}, \Lambda_P, \Lambda_T, \Lambda_{PT}\}$ as shown in Figure 2.3. Λ_P is a *parity* transformation that flips the sign of the determinant $(+ \leftrightarrow -)$, Λ_T is *time reversal* that flips the sign of $\Lambda^0{}_0$ $(\uparrow \leftrightarrow \downarrow)$ and Λ_{PT} is this combined operation.

The transformations we will be concerned with should keep spacetime in the same orientation for physicality (so space is not reflected and time continues running forward). This makes sense because one cannot physically reflect matter in 3D space or reverse time[9]. For this reason we choose to study the **restricted** Lorentz group $\mathrm{L}_+^\uparrow$ with elements satisfying $\det \Lambda = +1$ and $\Lambda^0{}_0 \geqslant +1$. We will come back to parity exchange and time reversal when reviewing **discrete** symmetries in Chapter 6.

The restricted Lorentz group can be decomposed into circular rotations J through space and hyperbolic rotations K in time. Circular rotations in 3D space alone would require us to study SO(3), but now we have time on our side, giving rise to the structure being SO(1, 3) (the keeping of the S for *special* in the orthogonal group is of course justified since $\det \Lambda = +1$). Moreover, the temporal constraint is denoted with a + on the group that tells us time runs forward. The restricted Lorentz group is

[9] If we could, the history of the universe might look somewhat different—though on a philosophical level I don't think we as the general public would actually be affected.

defined as the special orthogonal group in $1+3$ dimensions with a positive 00 component.

$$\mathrm{L}_+^{\uparrow} := \mathrm{SO}^+(1,3) = \{\Lambda \in \mathbb{R}^{4\times 4} \ : \ \Lambda^T \eta \Lambda = \eta, \ \det \Lambda = +1, \ \Lambda^0{}_0 \geqslant +1\}. \tag{2.3.5}$$

$$\begin{tikzcd}
\mathbb{1} \in \mathrm{L}_+^{\uparrow} \arrow[r, "\Lambda_P"] \arrow[d, "\Lambda_T"'] \arrow[dr, "\Lambda_P\Lambda_T"] & \mathrm{L}_-^{\uparrow} \ni \Lambda_P \arrow[d, "\Lambda_T"] \\
\Lambda_T \in \mathrm{L}_-^{\downarrow} \arrow[r, "\Lambda_P"'] & \mathrm{L}_+^{\downarrow} \ni -\mathbb{1}
\end{tikzcd}$$

FIGURE 2.3: The four disjointed components of the Lorentz group. Adapted from [98].

However, $\mathrm{SO}^+(1,3)$ is not the collection of all maps that preserve the invariant interval s^2—remember that's what we are setting out to find! This is because the Lorentz group keeps the origin fixed and is used to relate two observers when one reference point is set to be the origin. What we would like to consider instead involves translations too. Of course, displacements in both time and space keep the metric invariant since the metric does not depend on any coordinates. These extra four dimensions present us with the **Poincaré**[10] **group** which we can construct via the *semi-direct* product between Minkowski spacetime and the restricted Lorentz group:

$$\mathcal{P}(1,3) = \mathbb{R}^{1,3} \rtimes \mathrm{SO}^+(1,3). \tag{2.3.6}$$

Semi-direct as a product is like the standard direct product but one does not have to preserve Abelian group structures. The resulting group does not also need to be isomorphic to another group created by other products of the same two groups. In essence this means that two separate transformations in $\mathcal{P}(1,3)$ following one another can be defined as such. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{1,3}$ and $\Lambda_1, \Lambda_2 \in \mathrm{SO}^+(1,3)$, we define the elements $\{\Lambda_1 \,|\, \boldsymbol{a}\}, \{\Lambda_2 \,|\, \boldsymbol{b}\} \in \mathcal{P}(1,3)$ and define their product $\odot$ as:

$$\{\Lambda_1 \,|\, \boldsymbol{a}\} \odot \{\Lambda_2 \,|\, \boldsymbol{b}\} := \{\Lambda_1\Lambda_2 \,|\, \Lambda_1 \boldsymbol{b} + \boldsymbol{a}\}. \tag{2.3.7}$$

This product has a more intuitive picture when considering how it acts on spacetime coordinates x^μ:

$$\{\Lambda_2 \,|\, \boldsymbol{b}\} x^\mu = (\Lambda_2)^\mu{}_\nu x^\nu + b^\mu; \ \{\Lambda_1 \,|\, \boldsymbol{a}\} \odot \{\Lambda_2 \,|\, \boldsymbol{b}\} x^\mu = (\Lambda_1)^\mu{}_\rho \big[(\Lambda_2)^\rho{}_\nu x^\nu + b^\rho\big] + a^\mu, \tag{2.3.8}$$

which matches with what we defined above.

2.3.1 The Poincaré algebra

To understand what symmetries of the Poincaré group mean for us and, more fundamentally, particles, we turn to its algebra. We consider an infinitesimal transformation in $\mathcal{P}(1,3)$ where $\omega^\mu{}_\nu, \varepsilon^\mu \ll 1$ are taken to be infinitesimally small parameters:

$$\{\Lambda \,|\, \boldsymbol{a}\} x^\mu \approx (\delta^\mu{}_\nu + \omega^\mu{}_\nu)\, x^\nu + \varepsilon^\mu \quad \Longrightarrow \quad \delta x^\mu = \omega^\mu{}_\nu x^\nu + \varepsilon^\mu. \tag{2.3.9}$$

For Lorentz transformations on their own, we can substitute $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ into $\eta = \Lambda^T \eta \Lambda$ and observe:

$$\begin{aligned}
\eta_{\mu\nu} &= (\delta_\mu{}^\rho + \omega_\mu{}^\rho)\, \eta_{\rho\sigma}\, (\delta^\sigma{}_\nu + \omega^\sigma{}_\nu) \\
&= (\delta_\mu{}^\rho + \omega_\mu{}^\rho)\, (\delta_{\rho\nu} + \omega_{\rho\nu}) \\
&= \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} + \mathcal{O}(\omega^2),
\end{aligned} \tag{2.3.10}$$

keeping in mind that $\delta_{\nu\rho}$ can only be used for contraction when it has one upper and one lower index, and via the following (which was suppressed in the above calculation) the indices of one of the ω's are

[10] Henri Poincaré, 1854 - 1912, French, though the name of the group is credited to Minkowski.

exchanged:

$$\omega_\mu{}^\rho\delta_{\rho\nu} = \delta_{\nu\rho}\omega^\rho{}_\mu = \eta_{\rho\alpha}\eta^{\rho\beta}\delta_\nu{}^\alpha\omega_{\beta\mu} = \delta_\alpha{}^\beta\delta_\nu{}^\alpha\omega_{\beta\mu} = \omega_{\nu\mu}. \tag{2.3.11}$$

Since $\omega^\mu{}_\nu \ll 1$ we can ignore transformations at higher orders. To keep the above equation consistent we must have that the ω's are antisymmetric; $\omega_{\mu\nu} = -\omega_{\nu\mu}$. A real antisymmetric tensor in four dimensions has six free parameters, which justifies the dimension of the Lorentz group. Together with the four infinitesimal translations ε^μ we have ten free parameters that describe the Poincaré group $\mathcal{P}(1,3)$. In order to determine the group itself, we can exponentiate the algebra elements with six antisymmetric Lorentz generators $M^{\mu\nu}$ and four translation generators P^μ.

Imposing that the group elements of $\mathcal{P}(1,3)$ can be represented as exponentials of generators is a sensible thing to suggest; relativistic quantum field theories require unitary operators for invariant observables. What this means for us is that there *should* exist some unitary representation U associated to each Poincaré transformation. We will need some extra tools to fully understand how these will come about, but for now we can focus on just the Lorentz structure:

$$\Lambda = \exp\left\{\frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}\right\}, \tag{2.3.12}$$

where we must have $M^{\mu\nu} = -M^{\nu\mu}$ to match with the antisymmetry of $\omega_{\mu\nu}$. All of the transformations are completely real, which means the generators are themselves completely imaginary and therefore hermitian once the indices are lowered.

► It will be imperative to keep in mind that the matrix indices are suppressed in the above definitions; to be more concrete one should write the Lorentz generators as $(M^{\mu\nu})^\rho{}_\sigma$ so that

$$\eta_{\alpha\rho}(M^{\mu\nu})^\rho{}_\sigma = (M^{\mu\nu})_{\alpha\sigma} = -(M^{\mu\nu})_{\sigma\alpha}. \tag{2.3.13}$$

As for the Lie bracket, it is well-known that translations commute and we can automatically write $[P^\mu, P^\nu] = 0$. We can determine the other relations by considering how infinitesimal Lorentz transformations affect P^μ and $M^{\mu\nu}$ under the eyes of these two representations; one is vector-like and the other is operator-like, so transforms under the adjoint of (2.3.12). First, P^μ has the following dual transformation to first order in ω:

$$\begin{aligned} P^\sigma &\rightsquigarrow \left(\delta^\sigma{}_\nu + \frac{1}{2}(\omega^\sigma{}_\nu + \omega^\sigma{}_\nu)\right)P^\nu \\ &= P^\sigma + \frac{1}{2}\eta^{\sigma\mu}\left(\omega_{\mu\nu} - \omega_{\nu\mu}\right)P^\nu \\ &= P^\sigma + \frac{1}{2}\omega_{\mu\nu}\left(\eta^{\sigma\mu}P^\nu - \eta^{\sigma\nu}P^\mu\right). \end{aligned} \qquad \begin{aligned} P^\sigma &\rightsquigarrow \left(\mathbb{1} + \frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}\right)P^\sigma\left(\mathbb{1} - \frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \\ &\approx P^\sigma + \frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}P^\sigma - \frac{\mathrm{i}}{2}\omega_{\mu\nu}P^\sigma M^{\mu\nu} \\ &= P^\sigma - \frac{\mathrm{i}}{2}\omega_{\mu\nu}[P^\sigma, M^{\mu\nu}]. \end{aligned} \tag{2.3.14}$$

Note that in the final line for the vector-like transformation of P^σ we used the notion of dummy indices to swap μ and ν in the last term. Similarly for the vector-like components of $M^{\mu\nu}$,

$$\begin{aligned} M^{\mu\nu} &\rightsquigarrow \left(\delta^\mu{}_\alpha + \frac{1}{2}(\omega^\mu{}_\alpha + \omega^\mu{}_\alpha)\right)M^{\alpha\beta}\left(\delta_\beta{}^\nu + \frac{1}{2}(\omega_\beta{}^\nu + \omega_\beta{}^\nu)\right) \\ &\approx M^{\mu\nu} + \frac{1}{2}\eta^{\mu\rho}\left(\omega_{\rho\alpha} - \omega_{\alpha\rho}\right)M^{\alpha\nu} - \frac{1}{2}\eta^{\nu\sigma}\left(\omega_{\sigma\beta} - \omega_{\beta\sigma}\right)M^{\mu\beta} \\ &= M^{\mu\nu} + \frac{1}{2}\omega_{\rho\alpha}\left(\eta^{\mu\rho}M^{\alpha\nu} - \eta^{\mu\alpha}M^{\rho\nu}\right) + \frac{1}{2}\omega_{\beta\sigma}\left(\eta^{\nu\sigma}M^{\mu\beta} - \eta^{\nu\beta}M^{\mu\sigma}\right) \\ &= M^{\mu\nu} + \frac{1}{2}\omega_{\rho\sigma}\left(\eta^{\mu\rho}M^{\sigma\nu} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\rho\nu} - \eta^{\nu\rho}M^{\mu\sigma}\right), \end{aligned} \tag{2.3.15}$$

where to obtain the last line we made the switch $\alpha \to \sigma$ and $\beta \to \rho$ by virtue of dummy indices. Then:

$$\begin{aligned} M^{\mu\nu} \rightsquigarrow & \left(\mathbb{1} + \frac{\mathrm{i}}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right) M^{\mu\nu} \left(\mathbb{1} - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right) \\ &\approx M^{\mu\nu} + \frac{\mathrm{i}}{2}\omega_{\rho\sigma}M^{\rho\sigma}M^{\mu\nu} - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}M^{\mu\nu}M^{\rho\sigma} \\ &= M^{\mu\nu} - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}]. \end{aligned} \tag{2.3.16}$$

We can now collect this information together by equating each Lie bracket with the respective dual transformations.

The Poincaré algebra

For the six Lorentz generators $M^{\mu\nu}$ and four translation generators P^μ of $\mathcal{P}(1,3)$, we have the **Poincaré algebra**:

$$\begin{aligned} [P^\mu, P^\nu] &= 0; \\ [P^\sigma, M^{\mu\nu}] &= \mathrm{i}\left(\eta^{\sigma\nu}P^\mu - \eta^{\sigma\mu}P^\nu\right); \\ [M^{\mu\nu}, M^{\rho\sigma}] &= \mathrm{i}\left(\eta^{\mu\rho}M^{\sigma\nu} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\rho\nu} - \eta^{\nu\rho}M^{\mu\sigma}\right). \end{aligned} \tag{2.3.17}$$

Using this algebra we can decompose the $M^{\mu\nu}$'s into their spatial and temporal part and verify that calling them *rotations* and *boosts* was correct. If we define, for $1 \leqslant a, b, c \leqslant 3$,

$$J_a = \frac{1}{2}\varepsilon_{abc}M_{bc}, \quad K_a = M_{0a}, \tag{2.3.18}$$

then we calculate the following Lie brackets, noting that the product of totally antisymmetric tensors ε_{abc} gives twice the product of the sign of their permutations:

$$\begin{aligned} [J_a, J_b] &= \frac{1}{2}\varepsilon_{abc}\varepsilon_{bca}[M_{bc}, M_{ca}] \\ &= \mathrm{i}\left(\delta_{bc}M_{ac} + \delta_{ca}M_{bc} - \delta_{ba}M_{cc} - \delta_{cc}M_{ba}\right) \\ &= \mathrm{i}M_{ab} = \mathrm{i}\varepsilon_{abc}J_c. \end{aligned} \tag{2.3.19}$$

This is the algebra for spatial rotations, $\{J_a, J_b, J_c\} \in \mathfrak{so}(3)$. The other two follow similarly. Collating the information together we present the algebra for Lorentz generators J, K:

$$[J_a, J_b] = \mathrm{i}\varepsilon_{abc}J_c, \quad [K_a, K_b] = -\mathrm{i}\varepsilon_{abc}J_c, \quad [J_a, K_b] = \mathrm{i}\varepsilon_{abc}K_c. \tag{2.3.20}$$

The 4×4 matrix representation of these generators is given by the following, labelled by the axis they act on in 3D space:

$$\begin{gathered} J_x = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \; J_y = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; J_z = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ K_x = \mathrm{i}\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; K_y = \mathrm{i}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; K_z = \mathrm{i}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \end{gathered} \tag{2.3.21}$$

Observe that the rotation generators J are hermitian, matching with what we know about $\mathfrak{su}(2)$ generators. The boost generators K are anti-hermitian.

2.3.2 Conserved quantities

The main postulate from this chapter is that by its very construction, demanding that a physical system has invariance under Poincaré transformations means it must have symmetries in $M^{\mu\nu}$ and P^{μ}; the system is invariant in Lorentz transformations and translations. Symmetries are important, as highlighted in the introduction, because these come hand in hand with **conservation laws** of that physical system. We have studied and well-understood in previous courses that there exists a $1:1$ correspondence between the two, thanks to Noether's[11] theorem.

Consider just the translation generator P^{μ}. It can be split into its purely temporal and spatial components by realising time translations are generated by P^0 and space translations are generated by P^i. We label them in this way because these generators are canonically related to the 4-momentum p_{μ} of a given system, which is the relativistic way of capturing energy E and 3-momentum $\mathbf{p}$ in the same object. This tells us that P^0 as a generator is canonically intertwined with energy and must therefore correspond to the **Hamiltonian** $\mathcal{H}$ of a system—and we know that generators commuting with the Hamiltonian imply the very conservation laws we seek.

► Using some insights from Section 6.2 of [19], we know from Heisenberg's[12] picture of quantum mechanics that a system has symmetry with respect to an operator if the operator keeps the Hamiltonian $\hat{\mathcal{H}}$ invariant. That is, the operator does not depend explicitly on time. In the case of translations,

$$\hat{T}^{\dagger}\hat{\mathcal{H}}\hat{T} = \hat{\mathcal{H}}, \tag{2.3.22}$$

for some translation operator $\hat{T}$. We can rearrange the above to get $[\hat{\mathcal{H}}, \hat{T}] = 0$ if $\hat{T}$ is a unitary operator—this requires the generators of such operators to be hermitian. Then for continuous translations generated by the momentum operator $\hat{p}$, one can use infinitesimal translations and show

$$\hat{T} = \mathrm{e}^{-\mathrm{i}\delta\hat{p}} \approx \mathbb{1} - \mathrm{i}\delta\hat{p} \implies [\hat{\mathcal{H}}, \hat{p}] = 0, \tag{2.3.23}$$

and we interpret this as the **conservation of momentum** due to Ehrenfest's[13] theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle = \mathrm{i}\langle[\hat{\mathcal{H}}, \hat{p}]\rangle = 0. \tag{2.3.24}$$

Using the Poincaré algebra (2.3.17), any generator that commutes with $P^0 = \mathcal{H}$ must imply a conservation law. We have:

$$\begin{aligned} [P^0, P^{\mu}] = 0 &\implies \text{conservation of energy } E \text{ and 3-momentum } \mathbf{p}; \\ [P^0, J^a] = 0 &\implies \text{conservation of angular momentum } \mathbf{L}. \end{aligned} \tag{2.3.25}$$

The first is determined trivially as all translation generators commute. The second arises by asking for what values of μ, ν does $[P^0, M^{\mu\nu}] = 0$? From (2.3.18) this must imply $\eta^{0\nu}P^{\mu} = \eta^{0\mu}P^{\nu}$ for all μ, ν, which can only be the case when both sides of the equality are zero. This occurs when $\eta^{0\mu}$ acts in the spatial direction, so $1 \leqslant \mu, \nu \leqslant 3$ and we are thus left with the rotation generators J_a which are canonically related to angular momentum. This could also have been derived knowing that P^{μ} and J^a are hermitian generators and gives an intuitive reason as to why there is no conservation law associated with the K's; they are not hermitian generators, and is the reason we will not label particle states with any label to do with boosts. More of that to come in the next section.

The symmetries of spacetime give rise to the conservation of energy, 3-momentum and angular momentum.

[11] Emmy Noether, 1882 - 1935, German.
[12] Werner Heisenberg, 1901 - 1976, German, Nobel Prize 1932.
[13] Paul Ehrenfest, 1880 - 1933, Austrian.

2.4 Particles as representations

As highlighted in the introduction to this chapter we are aiming to find a way to represent elementary particles using algebraic structures. To use the word 'elementary' we mean that the representations, whatever they are, must be irreducible (if not, they could be decomposed into irreducible representations, saying that there exist smaller, more fundamental objects transforming in this way). The question is then how would we *class* or *categorise* particles at a fundamental level. We will see that the answer is by their **spin** and whether or not they have **mass**.

Furthermore, since these particles exist at a quantum level, whose behaviour is governed by unitary transformations, the representations of particles must be unitary. We begin with discussing the irreducible representations of the Lorentz group and then unitary irreducible representations of the Poincaré group. What follows uses arguments from Section 2.3 of [45], Section 4.5 of [43] and Sections 10.3, 10.4 of [28].

2.4.1 Irreducible representations of the Lorentz group

Recall the algebra of the restricted Lorentz group given by (2.3.20):

$$[J_a, J_b] = \mathrm{i}\varepsilon_{abc}J_c, \quad [K_a, K_b] = -\mathrm{i}\varepsilon_{abc}J_c, \quad [J_a, K_b] = \mathrm{i}\varepsilon_{abc}K_c. \tag{2.4.1}$$

These commutation relations tell us that rotations and boosts are linked and moreover, the representations are reducible. This can be seen when we define new generators A, B defined as

$$A_i = \frac{1}{2}(J_i + \mathrm{i}K_i), \quad B_i = \frac{1}{2}(J_i - \mathrm{i}K_i) \quad \Longrightarrow \quad (A_i)^\dagger = B_i. \tag{2.4.2}$$

Their commutation relations are given by:

$$\begin{aligned}
[A_a, A_b] &= \frac{1}{2}\left([J_a, J_b] - [K_a, K_b] + \mathrm{i}[J_a, K_b] + \mathrm{i}[K_a, J_b]\right) \\
&= \frac{1}{2}\left(2\mathrm{i}\varepsilon_{abc}J_c - 2\varepsilon_{abc}K_c\right) \\
&= \mathrm{i}\varepsilon_{abc}\left(J_c + \mathrm{i}K_c\right) = \mathrm{i}\varepsilon_{abc}A_c;
\end{aligned} \tag{2.4.3}$$

$$[B_a, B_b] = ([A_b, A_a])^\dagger = \mathrm{i}\varepsilon_{abc}B_c; \quad [A_a, B_b] = [A_a, (A_b)^\dagger] = 0.$$

The Lorentz group therefore has an algebra equivalent to two disconnected subalgebras (since the A, B commute), and it is easy to convince ourselves that these are $\mathfrak{su}(2)$ algebras. We therefore have the equivalence

$$\mathfrak{so}^+(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{2.4.4}$$

Why is this important? It tells us that to get irreducible representations of the Lorentz group we really just need irreducible representations of $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$. These have been discussed in previous courses regarding quantum mechanics, so we will go through only a brief recap.

The irreducible representations of SU(2) characterise the spin of elementary particles. To follow the same notation as in many quantum mechanics textbooks (for now), the chosen generators of SU(2) we denote by J_i and their commutator is as usual. Spin states of particles are denoted by $|j, j_z\rangle$, where j labels the largest 'allowed' spin of a particle and j_z labels the specific spin of a particle by acting on these states in the z-direction.

► Note that we choose the z-direction by mere convention; we seek eigenvalues for obervable quantities, which requires diagonalisation of our operator matrices. Even though we have called the operator matrices J_i we know that their 2×2 representation is given by Pauli's matrices. It is by convention that Pauli's third matrix σ_3 is diagonal, so we take that as our third spatial direction so that no extra work needs to be done to find its eigenvalues.

A particle's specific spin value can be measured with J_z, but we have to work harder to determine where we measure total allowed spin from. From the known spin-1/2 representation of SU(2) one constructs *ladder operators* that can take us up and down the different spin values, defined so that the raising operator annihilates the state of highest spin and vice versa for the lowering operator. This can then be naturally extended to constructing spin-j ladder operators in which to define irreducible spin-j representations of SU(2). This brings about defining the *Casimir*[14] *operator* $J^2 = J_i J_i$.

The beauty of this operator is that it commutes with all generators in the algebra while not actually being part of the algebra itself. The reason for this is because Lie algebras come equipped with a Lie bracket, meaning that is the only thing an element in the algebra 'understands', so to speak. Since a Casimir is a Lorentz scalar it often involves squaring elements; in particular it does not involve the Lie bracket, so cannot live inside the algebra. Instead, it lives inside a **universal enveloping algebra**. More information about these can be found in the bibliography [33, 103].

Moreover, because J^2 commutes with every element in $\mathfrak{so}(3)$ we say that the operators J^2, J_z can be *diagonalised simultaneously*, which is what really allows us to measure total allowed spin j. We can also use Schur's Lemma (see Theorem 1.4 in [43]) to our advantage: $J^2 \propto \mathbb{1}$ for a given representation. That is, the Casimir operator is an eigenvalue operator with the same eigenvalue for each state in the representation—J^2 acting on any spin state will provide us with a label for spin. In the case of SU(2) representations,

$$J^2|j, j_z\rangle = j(j+1)|j, j_z\rangle, \quad J_z|j, j_z\rangle = j_z|j, j_z\rangle. \tag{2.4.5}$$

It then follows that in order to define particle states we must label them with eigenvalues of Casimir operators of our symmetry group. With this, we can determine what the irreducible representations of $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$ are. If we define the ladder operators (taking inspiration from [89]):

$$\begin{aligned} A_+ &= \mathrm{i}A_x - A_y, \quad A_- = \mathrm{i}A_x + A_y, \quad A_{\text{eigen}} = \mathrm{i}A_z; \\ B_+ &= \mathrm{i}B_x - B_y, \quad B_- = \mathrm{i}B_x + B_y, \quad B_{\text{eigen}} = \mathrm{i}B_z, \end{aligned} \tag{2.4.6}$$

we can create states with total allowed spin determined by the Casimirs J_A^2, J_B^2. These irreducible representations can be labelled by the spins (j_A, j_B) and will automatically be irreducible representations of the Lorentz group, which is what we were looking for. We will return the specifics of these operators and what (j_A, j_B) means in terms of objects that we are already familiar with in Section 2.5.

For now, one notes that we cannot go further than this in an attempt to properly define particle states. Physical observables, as we know, require unitarity. Unitary operators must have hermitian generators but we know from (2.4.2) that the A, B are not hermitian. This is also reflected in the fact that $\mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$ is clearly a compact group but the Lorentz group is not. So these irreducible representations cannot correspond to physical states. For those, we turn to the Poincaré group.

2.4.2 Unitary irreducible representations of the Poincaré group

Following on from our above discussion, for the case of particles invariant under spacetime transformations we must be able to define particle states using the eigenvalues of Casimir operators of $\mathcal{P}(1,3)$. There are two of these. The first, a rather simple case, is formed by the translation generator P^μ. We know by Noether's theorem these are canonically related to momenta p_μ. This means that exponentiating the translation generators alone creates eigenvalue operators for 4-momentum

[14] Hendrik Casimir, 1909 - 2000, Dutch.

eigenstates $|p\rangle$. More explicitly, for some translation $T(a^\mu) \in \mathbb{R}^{1,3}$,

$$P_\mu|p\rangle := p_\mu|p\rangle \implies T(a^\mu)|p\rangle = \mathrm{e}^{\mathrm{i}a^\mu p_\mu}|p\rangle. \tag{2.4.7}$$

We are inclined to write down $P^2 = P^\mu P_\mu$ as our first Casimir of the Poincaré group; it is a Lorentz-invariant quantity and commutes with P^μ, clearly, and also with $M^{\mu\nu}$. Using (2.3.17),

$$\begin{aligned}
[P^\sigma P_\sigma, M^{\mu\nu}] &= \mathrm{i}\left(\eta^{\sigma\mu}P^\nu - \eta^{\sigma\nu}P^\mu\right)P_\sigma - \mathrm{i}P^\sigma\eta_{\rho\sigma}\left(\eta^{\rho\mu}P^\nu - \eta^{\rho\nu}P^\mu\right) \\
&= \mathrm{i}\left(P^\nu P^\mu - P^\mu P^\nu\right) - \mathrm{i}\left(P^\mu P^\nu - P^\nu P^\mu\right) \\
&= 2\mathrm{i}[P^\nu, P^\mu] = 0.
\end{aligned} \tag{2.4.8}$$

The eigenvalues one obtains from acting with P^2 on a state are

$$P^\mu P_\mu|p\rangle = p^\mu p_\mu|p\rangle = m^2|p\rangle, \tag{2.4.9}$$

knowing that the 4-momentum of a particle squared is nothing but its mass squared with the speed of light unit-normalised. Hence one of the labels for particle states must be their mass m. Now, for our second spacetime Casimir we know we cannot use $(M_{\mu\nu})^2$—it will not commute with $M^{\mu\nu}$. Another approach might be to use both generators. For this to work we need a Lorentz scalar that includes some product of $P^\sigma, M^{\mu\nu}$. We turn to the *Pauli–Lubanksi*[15] *vector*:

$$W_\mu := \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}, \tag{2.4.10}$$

developed by Lubański in several papers discussing elementary particles of any spin [72, 73]. It is a good choice of operator because its square is a Lorentz scalar and commutes with both Poincaré generators. We note its following properties:

$$\begin{aligned}
W_\mu P^\mu &= 0, \qquad & [W_\mu, M_{\rho\sigma}] &= \mathrm{i}\left(\eta_{\mu\rho}W_\sigma - \eta_{\mu\sigma}W_\rho\right), \\
[W_\mu, P_\nu] &= 0, \qquad & [W_\mu, W_\nu] &= \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma.
\end{aligned} \tag{2.4.11}$$

Particle states defined with this operator may look like

$$W^\mu W_\mu|m, s; \lambda_i\rangle = f(m, s)|m, s; \lambda_i\rangle, \tag{2.4.12}$$

for some label s and eigenvalues f we are yet to find. Note the inclusion of the label m since we know mass must be an integral feature of a particle state. The labels λ_i are there to represent the operators we can diagonalise simultaneously. This is analogous to having $SU(2)$ rotation states labelled $|j, j_z\rangle$; J^2 and J_z have this property. Since W_μ commutes with P_μ they can also be diagonalised simultaneously and hence one of these labels is momenta p_μ.

Before we get to these eigenvalues of W^2 one might want to know what W_μ is actually describing—this will lead us to defining an induced representation. First we have to note that W_μ itself does not form any kind of algebra. This is because the commutator between two elements is quadratic on the right-hand side (2.4.11) and, as previously noted, Lie algebras only understand the Lie bracket. But we can consider the vector space formed by the eigenstates defined above (2.4.12) and observe they are also eigenstates of P_μ. We can therefore replace any P_μ by their eigenvalues when W_μ operates on the states and see that a Lie bracket *can* be formed:

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}p^\nu M^{\rho\sigma}, \quad [W_\mu, W_\nu] = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}p^\sigma W^\rho. \tag{2.4.13}$$

[15] Józef Lubański, 1914 - 1946, Polish.

This is an algebra that underpins the **Little group** or **stability group** of p^μ, $\mathcal{L}(p^\mu)$. It is a subgroup of $\mathrm{SO}^+(1,3)$ (since the generators are combinations of the $M^{\mu\nu}$'s) that leaves a given momenta invariant,

$$\mathcal{L}(\mathring{p}^\mu) := \{\Lambda \in \mathrm{SO}^+(1,3) \ : \ \Lambda^\mu{}_\nu \mathring{p}^\nu = \mathring{p}^\mu\}, \tag{2.4.14}$$

where we denote a specific momenta by $\mathring{p}^\mu$. Since this is formed from a Lie algebra the elements of the Little group are exponentials of $\mathrm{i}W_\mu$; the representations are unitary. These may be combined with the unitary representations for translations $T(a^\mu)$ (2.4.7) to form a subgroup of $\mathcal{P}(1,3)$, namely

$$\mathscr{P} := \mathcal{L}(\mathring{p}^\mu) \otimes \mathbb{R}^{1,3}, \tag{2.4.15}$$

by which representations are given by the eigenbasis vectors of $\mathring{P} := \mathrm{span}\{|\mathring{p}\rangle\}$. Now we are in a position to see how this almost fits together: we have unitary irreducible representations of $\mathscr{P}$ and we would like to use these to form those of $\mathcal{P}(1,3)$; this is called *inducing* a representation from a subgroup to its ambient group. Following a definition from Section 3.1 of [46], we begin by identifying a space $\mathring{P}$ and constructing a new representation formed by its left cosets with respect to the Poincaré group. For each $\{\Lambda \,|\, a\} \in \mathcal{P}(1,3)$, construct:

$$\mathcal{P}(1,3)/\mathring{P} := \{\Lambda \,|\, a\}\mathring{P} = \Big\{\{\Lambda \,|\, a\}\mathring{p} \ : \ \mathring{p} \in \mathring{P}\Big\}. \tag{2.4.16}$$

We then suggest that each Poincaré transformation can be decomposed into a transformation by some $\boldsymbol{L} \in \mathscr{P}$ followed by a different Poincaré transformation: $\{\Lambda \,|\, a\} = \{\Lambda' \,|\, a'\}\boldsymbol{L}$. This has the physical interpretation of identifying the parts of any Poincaré transformation that leave the 4-momentum invariant and performing those transformations first. Observe that this lets us rewrite our cosets as:

$$\{\Lambda \,|\, a\}\mathring{P} = \{\Lambda' \,|\, a'\}\boldsymbol{L}\mathring{P} = \{\Lambda' \,|\, a'\}\mathring{P}, \tag{2.4.17}$$

where we absorb $\boldsymbol{L}$ into $\mathring{P}$ because $\boldsymbol{L}$ will obviously keep $\mathring{p}^\mu$ invariant; it is the combination of a Little group element and a translation. What this means is that the left cosets don't actually depend on the representation of $\mathring{P}$ and instead only depend on the left coset

$$\mathcal{P}(1,3)/\mathscr{P} := \{\Lambda' \,|\, a'\}\mathscr{P}. \tag{2.4.18}$$

These cosets encapsulate all transformations $\boldsymbol{L}$ followed by a Poincaré transformation—but that is what we defined to be some other arbitrary Poincaré transformation. Hence (2.4.18) is itself the entire collection of Poincaré transformations: the Poicaré group! We have therefore induced a representation from $\mathscr{P} \subset \mathcal{P}(1,3)$ onto $\mathcal{P}(1,3)$. These representations are constructed to be both unitary and irreducible, which is what we sought after.

2.4.3 Wigner's classification

All of this is to say that since unitary irreducible representations of the Poincaré group exist, they *must* be categorised by eigenvalues of the Casimirs P^2, W^2. This was realised by Wigner[16] who aimed to categorise both particles and fields in the late 1930s [84]. We have already seen the eigenvalues of P^2 correspond to mass; the eigenvalues of W^2 can then be studied for a given class of momenta, a given class of mass, and there are two cases we discuss for physical interest, continuing with Section 2.3 of [45] and taking insights from Section 2.5 of Weinberg's book [30].

[16] Eugene Wigner, 1902 - 1995, Hungarian-American.

★ Massive particles

These are states whose momenta satisfy $P^\mu P_\mu = -m^2 < 0$ (chosen this way due to our sign convention). One can always choose a reference frame such that the 4-momentum is given by

$$p^\mu = \begin{bmatrix} m & 0 & 0 & 0 \end{bmatrix}. \tag{2.4.19}$$

The Little group is the set of Lorentz transformations keeping this momenta fixed. For m to be unchanged we must act only on the spatial part of p_μ, which would leave us with the rotations $J \in \mathrm{SO}(3)$. Using (2.3.18), it follows that the Pauli–Lubanski vector in this case is

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}p^\nu M^{\rho\sigma} \implies W_0 = \varepsilon_{0abc}p^a M^{bc} = 0,\ W_a = \varepsilon_{a0bc}p^0 M^{bc} = mJ_a. \tag{2.4.20}$$

This gives us the Casimir operator

$$W^2 = m^2 J^2 \implies W^2|m, s\rangle = m^2 j(j+1)|m, s\rangle. \tag{2.4.21}$$

This allows us to replace s with a particle's spin $j \in \mathbb{Z}/2$—the notion of particle spin is built into this Casimir operator! The extra labels λ_i can, for massive particles, be translated to the particle's momenta p_μ and the component of spin in the z-direction j_z.

Massive one-particle state

We *define* an elementary particle of nonzero mass m and spin j to be the one-particle state:

$$|m, j; p_\mu, j_z\rangle, \tag{2.4.22}$$

which is a **unitary irreducible representation of the Poincaré group**.

These must be finite-dimensional representations for fixed 4-momentum since the dimension of a spin-j representation is $2j+1$ [19, p. 157-160].

★ Massless particles

These particles are such that $P^\mu P_\mu = 0$. Once again, a reference frame can be chosen such that the momenta is given by

$$p^\mu = \begin{bmatrix} E & 0 & 0 & E \end{bmatrix}. \tag{2.4.23}$$

While a state has no mass it may still contain energy, which is what we have denoted by E; photons are a clear example of this. The Pauli–Lubanski vector reads

$$\begin{aligned} W_\mu &= E\begin{bmatrix} -M^{12} & M^{23} - M^{02} & M^{31} + M^{01} & M^{12} \end{bmatrix} \\ &= E\begin{bmatrix} -J_z & J_x - K_y & J_y + K_x & J_z \end{bmatrix}. \end{aligned} \tag{2.4.24}$$

Here, the Little group is a little tricker to figure out. Consider its algebra formed in this case:

$$[W_\mu, W_\nu] = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}p^\sigma W^\rho = \mathrm{i}E\left(\varepsilon_{\mu\nu\rho 0} + \varepsilon_{\mu\nu\rho 3}\right)W^\rho. \tag{2.4.25}$$

Then

$$[W_1, W_2] = \mathrm{i}EW^3 - \mathrm{i}EW^0 = 0, \quad [W_3, W_1] = \mathrm{i}EW_2, \quad [W_3, W_2] = -\mathrm{i}EW_1. \tag{2.4.26}$$

This is precisely the algebra for translations along the W_1, W_2 directions and rotations about the W_3-axis. We call this the **Euclidean group** $\mathrm{E}(2)$ in two dimensions because in an abstract W space, this would keep the Euclidean distance $(W_1)^2 + (W_2)^2$ invariant. A strange phenomenon occurs here, however. Since the Little group for the massive representation was $\mathrm{SO}(3)$ we know we had finitely many unitary representations labelled with half-integers, but E(2) has infinitely many unitary representations. We can understand this by realising that W_1, W_2 commute and can thus be diagonalized simultaneously; let their eigenvalues be w_1, w_2, respectively. Then the Casimir W^2 acting on a state would look like

$$W^2|0,s\rangle = (w_1^2 + w_2^2)|0,s\rangle := s^2|0,s\rangle. \tag{2.4.27}$$

We have decided to define s in this way to realise that w_1, w_2, s as parameters trace out a circle:

$$w_1 = s\cos\vartheta, \quad w_2 = s\sin\vartheta. \tag{2.4.28}$$

Since sine and cosine are continuous functions there are an uncountably infinite number of values that w_1, w_2 could be, all between -1 and $+1$. We therefore label states with the continuous parameter ϑ so that our massless one-particle states are defined to be $|0, s; p_\mu, \vartheta\rangle$.

Weinberg highlights that even though this is a valid procedure of obtaining states, massless particles have not been observed to have any kind of continuous degree of freedom; ϑ should not be used to label states and we must find a workaround. We can impose that one-particle massless states are state vectors where $w_1, w_2 = 0 \implies s = 0$, so the only label left to use would come from W_3. This restricts us to a subgroup of the Little group E(2), namely SO(2): the generator of just rotations in two dimensions. The eigenvalue of W_3 used to distinguish these states we will call h, and one should note it is proportional to j_z. This insight (and knowing the restricted Little group is one of rotations) tells us that h gives the component of angular momentum in the z-direction—this is exactly the definition of spin but for a massless particle. We denote it *helicity*.

Massless one-particle state

We *define* a physical elementary particle of zero mass and helicity h to be the one-particle state:

$$|0,0;p_\mu,h\rangle := |p_\mu,h\rangle. \tag{2.4.29}$$

Because SO(2) is a subgroup of SO(3) we know that the values of h must be quantised: $h \in \mathbb{Z}/2$, and helicity can only come in the form of half-integers too. A key difference between these two types of representation (other than one is massless) is that there can only exist one state for each fixed 4-momentum p_μ. We know this because SO(2) $\cong$ U(1) only has one generator, so the irreducible representations are one-dimensional (see Theorem 1.33 in [46]).

As for the particles in the Standard Model, it turns out that they all arise from these *massless* representations, but that seems quite an absurd statement because we know from experimentation that there exist a lot of particles that are massive. We will see this more explicitly at the end of the chapter and the rest of the report will detail how we can fix this mathematically.

2.5 Spin-1/2 representations of the Lorentz group

After now understanding how particles of varying spin arise naturally in the language of representation theory, we can now zoom into a specific sector and discuss what it means for particles of spin-1/2 to transform under Lorentz transformations. The 4×4 matrix representation of $\mathrm{SO}^+(1,3)$ (2.3.21) is good in that it details exactly how scalars and 4-vectors transform with a Lorentz symmetry in mind. However, we know that particles are classified by their spin and their subsequent transformations require matrix irreps of varying dimensions. One particular irrep we shall describe is the 2×2 irrep that transforms particles exhibiting spin half. We construct these spin-1/2 irreps with inspired discussions from [89, 6].

When first learning about the equivalence between SO(3) and SU(2) we saw that each rotation on a 3-vector could be mimicked under the adjoint of an element in SU(2). Consider the Pauli vector

(2.1.6). By mere virtue of its construction we can transform this object under the adjoint of some $U \in \mathrm{SU}(2)$, which will be equivalent to a rotation $R \in \mathrm{SO}(3)$:

$$X \longmapsto U \begin{bmatrix} z & x - \mathrm{i}y \\ x + \mathrm{i}y & -z \end{bmatrix} U^\dagger \cong R\mathbf{x}. \tag{2.5.1}$$

Using the adjoint as our transformation is key in that it suggests there are two halves of the Pauli vector to consider; a 'left half' that transforms under U and a 'right half' that transforms under $U^\dagger$. In some sense, a right-half transformation would be the dual of the left half. Let us run with this idea and demand that the factorisation of a Pauli vector into the tensor product of smaller objects is indeed possible:

$$\begin{bmatrix} z & x - \mathrm{i}y \\ x + \mathrm{i}y & -z \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \otimes \begin{bmatrix} \xi^{1*} & \xi^{2*}. \end{bmatrix} \tag{2.5.2}$$

The determinant of the tensor product is also zero, which implies this kind of factoring may only be done for Pauli vectors with zero determinant:

$$\det X = -z^2 - y^2 - x^2 = 0. \tag{2.5.3}$$

A **Pauli spinor** is therefore a two-component object which transforms under the $\underline{\mathbf{2}}$ of SU(2):

$$\begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \longmapsto U \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}, \; \begin{bmatrix} \xi^{1*} & \xi^{2*} \end{bmatrix} \longmapsto \begin{bmatrix} \xi^{1*} & \xi^{2*} \end{bmatrix} U^\dagger. \tag{2.5.4}$$

Though this decomposition is credible in a mathematical sense, the physical interpretation of $\det X = 0$ means that $\mathbf{x}$ are vectors of zero length. This is obviously a non-physical phenomenon! However, through the eyes of special relativity this may be allowed; a 4-vector with no spatial length may describe spacelike or null vectors depending on the size of its temporal component. This motivates attempting to write 4-vectors as in 2×2 matrix form. With the help of a fourth Pauli-like matrix $\sigma_0 := \mathbb{1}$ we construct a **Weyl vector**:

$$x^\mu \sigma_\mu = \begin{bmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{bmatrix} =: W. \tag{2.5.5}$$

If a spinorial decomposition exists, we must have:

$$\begin{bmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} \otimes \begin{bmatrix} \psi^{1*} & \psi^{2*} \end{bmatrix}. \tag{2.5.6}$$

As above, the determinant of the tensor product is 0. This imposes the constraint:

$$\det W = \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 = 0. \tag{2.5.7}$$

This is nothing but the invariant interval of spacetime s^2. Spinor decompositions in this way therefore may only be allowed for null 4-vectors—but how do they transform? In spacetime, 4-vectors transform under $\mathrm{SO}(1,3)$ but this is not a compact group—this was obvious from our analysis at the beginning of Section 2.3. It is then sensible to suggest that we can transform W under the adjoint of some compact group whereby one spinor transforms with a 'left' element and its dual with a 'right' element. There was no proper distinction between these type of transformations with Pauli spinors, but the following discussion will make this notion more concrete. Suppose this compact group has elements L. Then

$$W \longmapsto LWL^\dagger \implies |L| \cdot |L^\dagger| = 1 \text{ to preserve } s^2. \tag{2.5.8}$$

So L must belong to the group of 2×2 complex matrices with unit determinant—the 2×2 **special linear complex matrices**:

$$\mathrm{SL}(2,\mathbb{C}) := \{L \in \mathbb{C}^{2\times 2} \ : \ |\det L| = 1\}, \tag{2.5.9}$$

For every Lorentz transformation on a 4-vector, there exist two $\mathrm{SL}(2,\mathbb{C})$ transformations that do the same job:

$$W \longmapsto LWL^\dagger = (-L)W(-L)^\dagger. \tag{2.5.10}$$

It is in this sense that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{SO}^+(1,3)$ just like $\mathrm{SU}(2)$ is the double cover of $\mathrm{SO}(3)$. We should then be able to identify their algebras as different representations of the same thing. Indeed, if we attempt to find a basis for 2×2 traceless ($|\det L| = 1$) complex matrices we find the following six elements:

$$\begin{aligned} \sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \sigma_2\sigma_3 &= \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \ \sigma_3\sigma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \sigma_1\sigma_2 = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}. \end{aligned} \tag{2.5.11}$$

One should recognise these generators as the three vectors and three bivectors from the Clifford algebra Cl_3 we saw in Figure 2.2 which is why they are labelled in such a way. More importantly, though, is that the dimensions of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}^+(1,3)$ match up. If we were to compare each of the three generators to the J's and K's (2.3.21), note that the vectors are hermitian so must correspond to the rotations J. The bivectors are anti-hermitian and therefore correspond to boosts K. In order for their Lie brackets to match up we impose the following renormalisation by 1/2 with a clear naming convention:

$$\begin{aligned} \mathrm{j}_x &= \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathrm{j}_y = \frac{1}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \ \mathrm{j}_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \mathrm{k}_x &= \frac{1}{2}\begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \ \mathrm{k}_y = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \mathrm{k}_z = \frac{1}{2}\begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}. \end{aligned} \tag{2.5.12}$$

Then

$$[\mathrm{j}_a, \mathrm{j}_b] = \mathrm{i}\varepsilon_{abc}\mathrm{j}_c, \ [\mathrm{k}_a, \mathrm{k}_b] = -\mathrm{i}\varepsilon_{abc}\mathrm{j}_c, \ [\mathrm{j}_a, \mathrm{k}_b] = \mathrm{i}\varepsilon_{abc}\mathrm{k}_c. \tag{2.5.13}$$

Exponentiating the renormalised vectors with some parameter ϑ gives us 2×2 rotation matrices that act on Pauli spinors with an angle of $\vartheta/2$. This comes for free when moving to $\mathfrak{sl}(2,\mathbb{C})$ since the j's correspond to the same rotations. In a similar fashion, exponentiating the 2×2 boost generators k_i with some parameter ϕ gives

$$\begin{aligned} \exp\{\mathrm{i}\phi\,\mathrm{k}_x\} &= \cosh(\phi/2)\cdot \mathbb{1}_2 - \sinh(\phi/2)\cdot\sigma_1 = \begin{bmatrix} \cosh(\phi/2) & -\sinh(\phi/2) \\ -\sinh(\phi/2) & \cosh(\phi/2) \end{bmatrix}; \\ \exp\{\mathrm{i}\phi\,\mathrm{k}_y\} &= \cosh(\phi/2)\cdot \mathbb{1}_2 - \sinh(\phi/2)\cdot\sigma_2 = \begin{bmatrix} \cosh(\phi/2) & \mathrm{i}\cdot\sinh(\phi/2) \\ -\mathrm{i}\cdot\sinh(\phi/2) & \cosh(\phi/2) \end{bmatrix}; \\ \exp\{\mathrm{i}\phi\,\mathrm{k}_z\} &= \cosh(\phi/2)\cdot \mathbb{1}_2 - \sinh(\phi/2)\cdot\sigma_3 = \begin{bmatrix} \cosh(\phi/2) - \sinh(\phi/2) & 0 \\ 0 & \cosh(\phi/2) + \sinh(\phi/2) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{-\phi/2} & 0 \\ 0 & \mathrm{e}^{\phi/2} \end{bmatrix}, \end{aligned} \tag{2.5.14}$$

recalling that $\sigma_a\sigma_b = i\sigma_c$ for an even permutation of a, b, c, so we are simply exponentiating $\{-\phi\cdot\sigma_c/2\}$. These are 2×2 boost matrices that must act on **Weyl spinors**. We now have a definitive way of transforming spinors in a spacetime-symmetric way:

$$\begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix} \longmapsto L\begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix}, \; \begin{bmatrix}\psi^{1*} & \psi^{2*}\end{bmatrix} \longmapsto \begin{bmatrix}\psi^{1*} & \psi^{2*}\end{bmatrix} L^\dagger. \tag{2.5.15}$$

While a null 4-vector Lorentz boosts with some parameter ϕ, the corresponding spinor and its dual both boost by $\phi/2$, which is good enough of a reason to call this the **spin-1/2 representation** of $\mathfrak{sl}(2,\mathbb{C})$. This had to be the case if the algebra of 4D Lorentz transformations $\mathfrak{so}^+(1,3)$ were to be isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. The spin-1/2 representation of 4D Lorentz transformations is therefore given by $\mathrm{SL}(2,\mathbb{C})$.

Additionally, the fact that we would be dealing with null 4-vectors tells us that any particle a Weyl spinor describes must all have lightlike motion and therefore be **massless**! This is an astonishing fact because experimentally all particles with spin-1/2 have mass. Either we must find a different representation of spin-1/2 particles or find a mechanism that can *give* these particles a mass. It turns out we can, in fact, do both, and they are related phenomena. We begin this discussion below with the notion of chirality and return in Section 4.3 that provides us a way to give fields a mass.

2.5.1 Chirality

One thing to note in the Lie brackets (2.5.13) is that they are completely even in the k's; by exchanging signs on the boost generators k_i we have the exact same algebra but for a different set of matrices. We note this as being a fundamental difference between SU(2) and SL(2, ℂ)(2): for each tensor dimension N there exists one unique irreducible representation of SU(2) and *two* unique, disjointed irreducible representations of $\mathrm{SL}(2,\mathbb{C})$. This was hinted at earlier when we realised we could decompose $\mathrm{SO}^+(1,3)$ into two copies of SU(2) that we labelled with subscripts A and B (2.4.4). What we really should have done was label them with L and R since in the literature the two representations are denoted by the **left-chiral** and **right-chiral** representations of $\mathrm{SL}(2,\mathbb{C})$. We can easily see they are disjointed by considering what a parity transformation does to both a 4-vector and a Weyl vector:

$$\begin{bmatrix}x^0\\ x^1\\ x^2\\ x^3\end{bmatrix} \longmapsto \begin{bmatrix}x^0\\ -x^1\\ -x^2\\ -x^3\end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix}x^0+x^3 & x^1-\mathrm{i}x^2\\ x^1+\mathrm{i}x^2 & x^0-x^3\end{bmatrix} \longmapsto \begin{bmatrix}x^0-x^3 & -x^1+\mathrm{i}x^2\\ -x^1-\mathrm{i}x^2 & x^0+x^3\end{bmatrix}. \tag{2.5.16}$$

If we decompose the transformed Weyl vector $\overline{W}$ into its temporal and spatial parts, one observes that $x^\mu\sigma_\mu \longmapsto x^0\sigma_0 + x^i(-\sigma_i)$. We will denote this parity-exchanged Weyl vector as

$$\overline{W} := x^\mu\bar{\sigma}_\mu \;\text{ for }\; \bar{\sigma}_\mu := (\mathbb{1}, -\boldsymbol{\sigma}). \tag{2.5.17}$$

The parity exchange has, in essence, treated the Pauli matrices as spatial basis vectors, which makes sense in terms of Clifford algebras. One then asks what this means for our Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Since the vectors and bivectors of Cl_3 can be exchanged by multiplication by i $(=\sigma_1\sigma_2\sigma_3)$, the exponentials of the different generators are

$$\exp\{\mathrm{i}\phi\mathrm{j}_a\} = \exp\{-\phi\cdot\sigma_b\sigma_c/2\}, \quad \exp\{\mathrm{i}\phi\mathrm{k}_a\} = \exp\{-\phi\cdot\sigma_a/2\}. \tag{2.5.18}$$

After exchanging the sign on each σ_i, they cancel for rotations and reverse the direction of the boosts. This is equivalent to exchanging the sign on the boost generators, which is where we began.

► This should come as no surprise. If we rotate a basis vector, say, from $\mathbf{e}_1$ to $\mathbf{e}_2$, this is equivalent to rotating $-\mathbf{e}_1$ to $-\mathbf{e}_2$. However, a boost in the direction of $\mathbf{e}_1$ is a reverse boost in the direction of $-\mathbf{e}_1$. Rotations are parity-symmetric, boosts are not.

Many sources disagree on which naming convention is 'correct' for the left- vs right-chiral representations, just as with which metric signature is correct. Obviously neither one is more correct, it is a matter of taste. In what follows we will use the original defined basis (2.5.12) for the left-chiral representation which we denote $(1/2,0)$, and the generators with a negative sign in front of the k's will be used for the right-chiral representation $(0,1/2)$. We label them in this way to represent the spin values of particle states for each chrial representation, (j_L, j_R).

With this comes the concept of left-chiral and right-chiral Weyl spinors. For a Lorentz transformation $L \in \mathrm{SL}(2,\mathbb{C})$, we have the following actions on the two types of spinors:

$$\begin{aligned} &\text{left-chiral}, \qquad \Psi^\alpha = \begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix} \longmapsto L\begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix} = L^\alpha{}_\beta \Psi^\alpha; \\ &\text{right-chiral}, \qquad \bar{\boldsymbol{\chi}}_{\dot\alpha} = \begin{bmatrix}\bar\chi_{\dot 1}\\ \bar\chi_{\dot 2}\end{bmatrix} \longmapsto \bar L\begin{bmatrix}\bar\chi_{\dot 1}\\ \bar\chi_{\dot 2}\end{bmatrix} = \bar L_{\dot\alpha}{}^{\dot\beta}\bar{\boldsymbol{\chi}}_{\dot\beta}. \end{aligned} \tag{2.5.19}$$

$\bar L$ here is our notation for $(L^\dagger)^{-1}$, and it should make sense why this is the correct way for L to act on a right-chiral Weyl spinor: pure rotations are unitary transformations, so the dagger will cancel out the inverse on $(L^\dagger)^{-1}$ and will not change under parity exchange. Boosts, however, are hermitian transformations and so we leave ourselves with just an inverse transformation—a reversed boost.

Note also the use of the Greek indices $\alpha, \beta = 1,2$ and dotted indices that distinguishes between each representation. Although these are two-component objects they are not 2-vectors; they transform in a fundamentally different way.

2.5.2 Dirac spinors

To conclude this chapter, we finally use STA introduced in Section 2.2 to form **Dirac spinors** which is how we describe spin-1/2 particles in the Standard Model. To do this we consider the bivectors of $\mathrm{Cl}_{3,1}$:

$$\mathrm{Cl}_{3,1}^{(2)} = \{\gamma_0\gamma_1,\ \gamma_0\gamma_2,\ \gamma_0\gamma_3,\ \gamma_2\gamma_3,\ \gamma_3\gamma_1,\ \gamma_1\gamma_2\}. \tag{2.5.20}$$

To get a feel for what the bivectors represent we decompose them into the temporal and spatial sectors:

$$\gamma_0\gamma_a = 2\begin{bmatrix}\sigma_a & 0\\ 0 & \bar\sigma_a\end{bmatrix}; \quad \gamma_b\gamma_c = 2\mathrm{i}\varepsilon_{bca}\begin{bmatrix}\sigma_a & 0\\ 0 & \sigma_a\end{bmatrix}. \tag{2.5.21}$$

These matrices are entirely block diagonal, which means whatever they represent must be a reducible representation. But recall the special orthogonal Lie algebra relation (2.1.8) that told us the bivectors from any Clifford algebra have an isomorphism to the special orthogonal Lie algebra of that same space. $\mathrm{Cl}_{3,1}$ is defined on Minkowski spacetime, so $\mathrm{Cl}_{3,1}^{(2)} \cong \mathfrak{so}(1,3)$. This is, of course, the Lie algebra of the Lorentz group with time pointing forward, but is also recognised as $\mathfrak{sl}(2,\mathbb{C})$. We have therefore found a reducible representation of $\mathrm{SL}(2,\mathbb{C})$; a reducible spin-1/2 representation of the Lorentz group. In order to match the Lie brackets exactly we first relabel the bivectors so that they become Lorentz generators:

$$\mathcal{J}_a := -\frac{\mathrm{i}}{8}\varepsilon_{abc}\gamma_b\gamma_c = \frac{1}{2}\begin{bmatrix}\sigma_a & 0\\ 0 & \sigma_a\end{bmatrix}, \quad \mathcal{K}_a := \frac{\mathrm{i}}{4}\gamma_0\gamma_a = \frac{\mathrm{i}}{2}\begin{bmatrix}\sigma_a & 0\\ 0 & \bar\sigma_a\end{bmatrix}, \tag{2.5.22}$$

which mimics the definition for vector (spin-1) generators (2.3.18). The Lie bracket can easily be checked since the top-left components are the spin-1/2 Lorentz generators in the left-chiral basis and the bottom-right are those in the right-chiral basis. We therefore have:

$$[\mathcal{J}_a, \mathcal{J}_b] = \mathrm{i}\varepsilon_{abc}\mathcal{J}_c, \quad [\mathcal{K}_a, \mathcal{K}_b] = -\mathrm{i}\varepsilon_{abc}\mathcal{J}_c, \quad [\mathcal{J}_a, \mathcal{K}_b] = \mathrm{i}\varepsilon_{abc}\mathcal{K}_c. \tag{2.5.23}$$

This is indeed a Lorentz algebra. It is then clear that when we exponentiate these generators they become block-diagonal Lorentz transformations; one block for Ψ^α and the other for $\bar{\boldsymbol{\chi}}_{\dot{\alpha}}$. This is interpreted as the direct sum of a left-chiral and right-chiral Lorentz transformation. We therefore define a **bispinor** or **Dirac spinor** to be the direct sum of two Weyl spinors. This will form a four-component spinor whose top half transforms under the left-chiral representation of the Lorentz group and bottom half that transforms under the right-chiral representation:

$$\Psi^{\alpha}{}_{\dot{\beta}} := \begin{bmatrix} \psi^1 \\ \psi^2 \\ \bar{\chi}_{\dot{1}} \\ \bar{\chi}_{\dot{2}} \end{bmatrix} \longmapsto \left[\begin{array}{c|c} L & 0 \\ \hline 0 & \bar{L} \end{array}\right] \begin{bmatrix} \psi^1 \\ \psi^2 \\ \bar{\chi}_{\dot{1}} \\ \bar{\chi}_{\dot{2}} \end{bmatrix} =: \left(\Lambda_{1/2}\right)^{\alpha}{}_{\alpha'}{}^{\dot{\beta}'}{}_{\dot{\beta}} \Psi^{\alpha'}{}_{\dot{\beta}'}, \tag{2.5.24}$$

where $\Lambda_{1/2} \in \mathrm{SL}(2,\mathbb{C})_L \oplus \mathrm{SL}(2,\mathbb{C})_R$. We then say that Ψ lives in the $(1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group. Just as above, although these are four-component objects, Dirac spinors transform in a completely different way to 4-vectors that transform according to $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$. We may extract the two Weyl spinors from a Dirac spinor by making use of the pseudoscalar γ_5:

$$P_L\Psi := \frac{1}{2}(1+\gamma_5)\Psi^{\alpha}{}_{\dot{\beta}} = \begin{bmatrix} \psi^1 \\ \psi^2 \\ 0 \\ 0 \end{bmatrix}, \quad P_R\Psi := \frac{1}{2}(1-\gamma_5)\Psi^{\alpha}{}_{\dot{\beta}} = \begin{bmatrix} 0 \\ 0 \\ \bar{\chi}_{\dot{1}} \\ \bar{\chi}_{\dot{2}} \end{bmatrix}. \tag{2.5.25}$$

The P_L, P_R are known as the *projection operators* for Dirac spinors, and two spinors under these operators are denoted ψ_L, ψ_R, respectively.

A closing thought for this chapter is perhaps wondering why we even consider Dirac spinors at all. Yes, they appear naturally in the theory of STA but does that mean they appear naturally in the universe? We mentioned at the beginning of Section 2.4 that elementary particles may only appear as irreducible representations, otherwise there would be no reason to call them elementary. Dirac spinors are manifestly reducible representations of the Lorentz group so one would logically argue that Dirac spinors do not describe elementary particles.

This is true for the massless case—as highlighted part way through this chapter, mathematically all of our spin-1/2 particles are massless, and so the Dirac equation describing these particles decouples into the two separate representations [47, p. 13]. But in the massive case, a phenomena known as **spontaneous symmetry breaking** occurs and mixes the two representations together and ensures the representation cannot be reduced. In Chapter 4 we understand what this symmetry breaking actually is, and Chapter 5 details this specifically with electroweak interactions. This is important to note because the electroweak interactions act on all spin-1/2 particles, and through spontaneous symmetry breaking we give these elementary particles a mass, making a Dirac spinor fundamental to the Standard Model.

3
The Standard Model is a Field Theory

We are really trying to bring together quantum mechanics and special relativity. So far this has brought us to taking spacetime invariance and insisting we could describe transformations as unitary operators which would in turn give us a definition of a particle. The fundamental building blocks of the universe arise naturally from this group-theoretic description. However, the Standard Model is not a theory of particles—it is a theory of **fields** [27, p. 527]. We must therefore change our point of view from particles and move to that of fields. This will bring about the introduction to **quantum field theory**, which we will see is a more consistent approach to unifying these two areas of mathematical physics that is fundamental for studying particle interactions specifically.

In this chapter we will define what a field is and bring out the Lagrangian[1] formalism that will allow us to start giving some structure to the Standard Model. In Section 3.1 we will give reason to studying quantum field theory and why it is in fact a *necessity* for particle physics. We then relate this concept to symmetry arguments in Section 3.2 and discuss some more symmetries that arise when quantum field theory is considered; these are **internal symmetries** that we hinted at in Chapter 2.

We will then state in this chapter a theorem that says the symmetries we have studied thus far are the *only* symmetries we have to consider to fully appreciate the Standard Model. They are of importance to us because these symmetries then help us build said Lagrangians. These 'mother' equations of physical systems encode information about conservation laws and the equations of motion for the fields inside it. Indeed, all of the work we did in the last chapter while deriving the Poincaré algebra was to say that in a field theory, mathematical objects built from obeying Poincaré invariance automatically belong in a Lagrangian that describe the behaviour of fields that flow through spacetime.

To conclude this chapter we work towards gauge theory in Section 4.3.1, since understanding this will be fundamental towards understanding particle interactions.

3.1 Quantum field theory

Quantum field theory (QFT) is an exceptional area of mathematical physics that, at the time of writing, has been developed and refined over the course of almost a century. Its purpose is to unify quantum mechanics with special relativity with a strong focus of encoding particle interactions. It therefore 'constitutes the essential background underlying the Standard Model', as said by Penrose [22, p. 656]. As QFT isn't the main focus of this report and only one of the concepts required to understand why the Standard Model is built like it is, we may follow a heuristic approach for its introduction and accept some basic definitions.

As the author, when learning QFT for myself I followed an Advanced Quantum Theory course led by Dr Nagy at Durham whose lecture notes [42] are unpublished. This introduction will follow the notes I took alongside some notes with references given as we go along, though the lecture notes themselves credit some details to Srednicki [27] and Peskin & Schroeder [23]. The reader is directed to the bibliography where there are a plethora of textbooks to study from if they are so inclined to learn more about the subject. We have the books above (including Chapters 24-26 of [22]), as well as the books by Weinberg [30] and Lancaster & Blundell [20].

[1] Joseph-Louis Lagrange, 1736 - 1813, Italian.

3.1.1 Why field theory?

One might naturally think that aiming for a theory of both special relativity and quantum mechanics will converge to some kind of relativistic quantum mechanics. This is indeed what we would expect from the previous chapter; the particle states we defined (2.4.22, 2.4.29) definitely had relativity in mind. But these were *single* particle states, and some attempts at finding equations of motion for these states proved impossible because it was in fact necessary to be constantly thinking about multi-particle states (see, eg, [27, p. 23-25]).

We can understand this by considering how energy changes its form in relativistic notation. We may write a non-relativistic particle's energy in relation to its mass m and momentum $\mathbf{p}$, namely via

$$E = \frac{|\mathbf{p}|^2}{2m}. \tag{3.1.1}$$

The relativistic correction to this equation is

$$E = \sqrt{|\mathbf{p}|^2c^2 + m^2c^4} \approx mc^2 + \frac{|\mathbf{p}|^2}{2m} + \mathcal{O}\left(|\mathbf{p}|^4\right), \tag{3.1.2}$$

where we have expanded this in power of $\mathbf{p}$ to second-order Taylor. We see that the nonrelativistic energy makes its appearance only when the particle has momentum; when it is moving. When a particle is not moving we unveil perhaps the most famous equation in the world: $E = mc^2$.

This is Einstein's[2] mass-energy relation, an afterthought of special relativity[3] introduced as part of his *annus mirabilis* papers published in 1905 [61], and astoundingly it predicts that a stationary particle still carries energy. But we could reverse our logic and note that if enough energy is supplied to a vacuum state then a particle can be *created*. In particular, if perhaps we supply an energy $E = 2mc^2$, a particle/antiparticle pair could be produced—already we see that multi-particle states must be considered.

How does this relate to quantum mechanics? In a quantum mechanical framework, energy observables will be eigenstates of energy. What this means is that energy will exist as a flurry of fluctuations and will only collapse into the eigenstates when measured (see, eg, [19, p. 96]); so one may describe any system with having energy ΔE. If indeed there are enough fluctuations in energy such that

$$\Delta E \approx 2mc^2, \tag{3.1.3}$$

then somewhat miraculously a particle/antiparticle pair could be produced in a sea of pure fluctuations. This points us towards the idea that states exist where the number of particles can spontaneously change. Consequently this contradicts quantum mechanics; Schrödinger's equation of motion for wavefunctions admits solutions for an *exact* number of particles, and we must be ready for the particle number to change at any point. A framework proponed by Dirac led to a theory of fields, where energy flows through and, when enough is supplied, creates particles and antiparticles at will.

Particles may then be considered as the **quanta** of fields; an excitation in the field, a "bundle of energy" [92], which is interpreted as the particle's mass via Einstein's theory.

3.1.2 Poincaré transformations on fields

In order to describe physics with fields we must ensure they transform consistently with what we have developed thus far. Our previous discussions have focused on coordinate transformations, whereas now

[2] Albert Einstein, 1879 - 1955, German, Nobel Prize 1921.

[3] Interestingly, the paper only specifies that the mass of a body emitting energy L will diminish by L/c^2, and does not state the result as we praise it today [95]. The result was widely-used in many subsequent papers, though it may have been Lorentz who first wrote it down in the form '$\varepsilon = Mc^2$' [21, p. 24].

we are dealing with *functions* of coordinates. This means that the generators of such transformations can no longer be matrices and must be **differential operators** that act on the function, and as one might expect the form of these operators will change with the representation of the Lorentz group. Let us begin with a scalar (spin-0) field $\phi(x)$ and recall that these Lorentz transformations are the trivial ones, so:

$$\phi(x) \longmapsto \phi'(x') = \phi'\left(\{\Lambda \,|\, a\}x\right) = \phi(x). \tag{3.1.4}$$

The analysis of Poincaré transformations can be split into the translations and Lorentz generators separately, following Section 2.6 of [42]. We first consider pure translations, so that

$$\phi'(x') = \phi'(x-a) = \phi(x) \implies \phi'(x) = \phi(x+a). \tag{3.1.5}$$

If there exists some differential operator Q such that $\phi'(x) = \exp\{\mathrm{i}a_\mu Q^\mu\}\phi(x)$, then we can expand this to first order and see:

$$\exp\{\mathrm{i}a_\mu Q^\mu\}\phi(x) \approx \phi(x) + \mathrm{i}a_\mu Q^\mu \phi(x) + \mathcal{O}(a^2). \tag{3.1.6}$$

Then we expand an infinitesimal version of (3.1.5), whereby $a_\mu \ll 1$:

$$\phi'(x) \approx \phi(x) + a^\mu \frac{\partial}{\partial x^\mu}\phi(x) + \mathcal{O}(a^2). \tag{3.1.7}$$

By equating the coefficients we see that for the differential operator to match we must have $Q^\mu = -\mathrm{i}\partial^\mu$. As for the Lorentz transformations, we now demand:

$$\phi'(x) = \phi'(\Lambda x) = \phi(x) \implies \phi'(x) = \phi(\Lambda^{-1}x). \tag{3.1.8}$$

We may set $\Lambda = \exp\{\mathrm{i}\omega_{\rho\sigma}M^{\rho\sigma}\}$ as usual. We then suppose that the expansion of $\phi(\Lambda^{-1}x)$ can be written in terms of some differential operator $L^{\rho\sigma}$ acting on $\phi(x)$. When this is expanded we have:

$$\phi'(x) = \exp\{\mathrm{i}\omega_{\rho\sigma}L^{\rho\sigma}\}\phi(x) \approx \phi(x) + \mathrm{i}\omega_{\rho\sigma}L^{\rho\sigma}\phi(x) + \mathcal{O}(\omega^2). \tag{3.1.9}$$

Further, the transformation on the coordinates themselves are written as $\Lambda^{-1}x = \exp\{-\mathrm{i}\omega_{\rho\sigma}M^{\rho\sigma}\}x$. If we suppose that the ω's are infinitesimal parameters, we expand as such:

$$\phi'(x) \approx \phi(x) - \mathrm{i}\omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu}\phi(x) + \mathcal{O}(\omega^2). \tag{3.1.10}$$

These should be one and the same thing, and to equate them we can figure out an explicit form of the M's knowing that $M_{\rho\sigma} \propto \gamma_\rho\gamma_\sigma$. In particular, we can use an argument similar to when we derived the $\mathrm{SO}(N)$ generators as (2.1.11) and see

$$(M^{\rho\sigma})^\mu{}_\nu = -\mathrm{i}(\eta^{\mu\rho}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu). \tag{3.1.11}$$

With this, we compare the coefficients of the two transformations and deduce

$$\begin{aligned} L^{\rho\sigma} &= -(M^{\rho\sigma})^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} \\ &= \mathrm{i}(\eta^{\mu\rho}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu)x^\nu\partial_\mu \\ &= \mathrm{i}\left(x^\sigma\partial^\rho - x^\rho\partial^\sigma\right). \end{aligned} \tag{3.1.12}$$

For our later analysis we must also consider how **spinor fields** and **vector fields** transform. The calculations simply follow suit, and will not be proven here, but the results are depicted below. We note that for vector and spinor fields, they each have an extra term in the Lorentz generator that is their associated matrix generator. While the differential part of the operators act to keep the coordinate transformations symmetric, we must remember that vector fields and spinor fields may also rotate/boost in their respective spaces because they are *themselves* a vector or a spinor. The matrix part of the operator ensures the transformation of the fields is in the correct manner.

Differential operator representations of the Poincaré generators

The Poincaré transformations acting on various fields require differential operators as their generators. Namely,

$$Q^\mu := -\mathrm{i}\partial^\mu \quad \text{for all translations,} \tag{3.1.13}$$

and the following for Lorentz transformations:

$$\begin{aligned} L^{\rho\sigma} &:= -\mathrm{i}\left(x^\rho\partial^\sigma - x^\sigma\partial^\rho\right) && \text{on } \textbf{scalar fields};\\ (L^{\rho\sigma}) &:= M^{\rho\sigma} - \mathrm{i}\left(x^\rho\partial^\sigma - x^\sigma\partial^\rho\right) && \text{on } \textbf{vector fields};\\ (L^{\rho\sigma}) &:= \frac{1}{4}\gamma^\rho\gamma^\sigma - \mathrm{i}\left(x^\rho\partial^\sigma - x^\sigma\partial^\rho\right) && \text{on } \textbf{spinor fields}. \end{aligned} \tag{3.1.14}$$

These will all evidently satisfy the Poincaré algebra (2.3.17).

3.1.3 The Lagrangian formalism

With field transformations under our belt we can now begin to build the equations that allow us to describe their motion and interactions. The most useful tool at hand will be the **Lagrangian**[4] $\mathscr{L}$: a scalar object that encodes conservation laws and determines equations of motion [102]. For any Lagrangian respecting the symmetries of spacetime we demand it has Poincaré invariance. That is, by considering the differential operators above, we suggest that a Lagrangian is a Lorentz scalar (for Lorentz invariance) that does not depend explicitly on coordinates x^μ (for translation invariance). It should therefore be the sum of Lorentz scalars constructed out of fields and their derivatives: $\mathscr{L} = \mathscr{L}[\Phi, \partial_\mu\Phi]$. We have some simple examples:

- A free, real scalar field $\phi(x) \sim |m, j = 0\rangle$ with some harmonic potential,

$$\mathscr{L}[\phi, \partial_\mu\phi] \;\propto\; \partial_\mu\phi\partial^\mu\phi + m^2\phi^2; \tag{3.1.15}$$

- An interacting complex scalar field $\phi(x)$,

$$\mathscr{L}[\phi, \phi^\dagger, \partial_\mu\phi, \partial_\mu\phi^\dagger] \;\propto\; \partial_\mu\phi^\dagger\partial^\mu\phi + m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2; \tag{3.1.16}$$

- A vector field $A_\mu \sim |m, j = 1\rangle$,

$$\mathscr{L}[A_\mu, \partial_\mu A_\nu] \;\propto\; (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + m^2 A_\mu A^\mu; \tag{3.1.17}$$

- A free (Dirac) spinor field $\Psi^\alpha{}_{\dot\beta} \sim |m, j = 1/2\rangle$,

$$\mathscr{L}[\Psi, \partial_\mu\Psi] \;\propto\; \overline{\Psi}\mathrm{i}\partial\!\!\!/\Psi - m^2\overline{\Psi}\Psi = \overline{\Psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m^2\right)\Psi. \tag{3.1.18}$$

► We will discuss each of these Lagrangians in turn and understand what they aim to describe later in this report. For the scalar fields see Section 4.1, the vector fields see Section 4.3 and for spinor

[4]Technically we will be discussing the Lagrangian *density*, whereby the real Lagrangian L will be this quantity integrated over space, but it is more convenient to simply write 'Lagrangian'.

fields see Section 5.3. Note we have also introduced Feynman's[5] slash notation in (3.1.18) to contract the derivative with the γ's to compactify notation.

A Lagrangian comes with its associated action $\mathcal{S}$ by integrating $\mathscr{L}$ over all of spacetime:

$$\mathcal{S}[\Phi, \partial_\mu \Phi] = \int_{\mathbb{R}^{1,3}} \mathscr{L}[\Phi, \partial_\mu \Phi] \, \mathrm{d}^4 x. \tag{3.1.19}$$

We obtain the Euler-Lagrange field equations of motion by a familiar procedure of minimising said action (see, eg, [23, p. 15-16]):

$$\delta \mathcal{S} = 0 \implies \frac{\partial \mathscr{L}}{\partial \Phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)}\right) = 0. \tag{3.1.20}$$

The final pieces of valuable information the Lagrangian gives us are a **Noether current** j^μ and a **Noether charge** Q. Whenever a Lagrangian exhibits a continuous symmetry with infinitesimal transformation $\Phi \rightsquigarrow \Phi + \delta\Phi$, one constructs

$$j^\mu := \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \delta\Phi, \quad Q := \int_{\mathbb{R}^3} j^0 \, \mathrm{d}^3 x = \int_{\mathbb{R}^3} \frac{\partial \mathscr{L}}{\partial(\partial_0 \Phi)} \delta\Phi \, \mathrm{d}^3 x. \tag{3.1.21}$$

The way we construct these is so that the 4-divergence of j^μ vanishes[6] and hence the Noether charge is manifestly invariant in time; it is a **conserved quantity**. For translations and Lorentz transformations we will end up with statements referring to conservation of energy, momentum and angular momentum as predicted earlier (2.3.25). There are more conservation laws to unpack when we consider internal symmetries, which is where we are headed now.

3.2 Internal symmetries

The symmetries of spacetime were very much a physical thing; Poincaré invariance is deep-rooted into our definition of particles and offers a way to express field transformsations. Though there is a subtlety in the difference between the term 'Poincaré invariance' in the particles vs fields framework. Namely, on particles we constructed coordinate transformations that left the Minkowski metric $\eta_{\mu\nu}$ unchanged. From the field point of view we now demand that Poincaré transformations leave the *action* invariant [45, p. 50]. If we simply took the latter as our definition of symmetries, then this opens up a new variety of symmetries to consider: *internal symmetries*.

It turns out that these are the only type of symmetries we are left to discover in the context of the Standard Model. Before we get to them, it will be useful to know how we combine them.

The Coleman-Mandula theorem

If the symmetry group G of an interacting quantum field theory contains a subgroup of the Poincaré group and some maximal group of internal symmetries, then it must factorise as

$$\mathrm{G} = \text{ 'Poincaré' } \otimes \text{ 'Internal' } \quad [57, 104, 49]. \tag{3.2.1}$$

In particular, for the Standard Model, we have the symmetry group to be

$$\mathrm{G}_{SM} = \mathcal{P}(1,3) \otimes (\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)). \tag{3.2.2}$$

[5] Richard Feynman, 1918 - 1988, American, Nobel Prize 1965.

[6] This is often referred to as a *continuity equation* as it relates how a variable changes over time with respect to change over space; $\partial_t j^0 = -\boldsymbol{\nabla} \cdot \mathbf{j}$ [31, p. 840].

This theorem was developed by Coleman and Mandula[7] in the context of scattering amplitudes and states that there is no nontrivial way to combine the spacetime symmetries with the internal ones; it must be via a simple direct product. The proof of this statement requires knowledge of scattering amplitudes, which goes beyond the purpose of this report, so the reader is directed to the original paper [57] and an article written by Mandula himself that also includes some interesting history towards the initial problem [9].

A reason to use the word 'internal' is that these transformations only act on the field itself and do not affect the coordinates. That is, a field may undergo some transformation that does not change its equations of motion and yet we would not have done anything to its position in spacetime. In what follows we only theorise their existance in nature, but they are, in fact, known about from experiments on elementary particles. In a sense *they describe the 'inner workings' of the forces of nature* [93].

We best describe these symmetries via examples. A simple first case is one we have already seen while studying quantum mechanics, that being the invariance of phase shifts for complex wavefunctions. It is mentioned throughout Griffiths' book that this phase "carries no physical significance" on our system [19, p. 32]. We know this because the wavefunction itself is unobservable, and most[8] physical predictions from this only depend on the probability amplitude, which is left unchanged by a phase shift. Mathematically speaking, a change of complex phase to a wavefunction ψ comes of the form

$$\psi \longmapsto \mathrm{e}^{\mathrm{i}\alpha}\psi, \quad \psi^\dagger \longmapsto \mathrm{e}^{-\mathrm{i}\alpha}\psi^\dagger \quad \Longrightarrow \quad |\psi|^2 = \psi^\dagger\psi \text{ is unchanged} \tag{3.2.3}$$

for some phase parameter α. We immediately recognise this as a U(1) transformation, so in particular the Schrödinger equation has an internal U(1) symmetry since it describes the time evolution of $|\psi|$.

An important example (that we will see the significance of in Section 4.1) is the Lagrangian given above for an interacting complex scalar field (3.1.16). It too has an internal U(1) symmetry and for this reason it aims to describe a charged particle. Furthermore, note we did not specify the dimension of the scalar field. If fields were to decide to transform under groups of higher dimension they better be unitary, so one could say a field $\boldsymbol{\phi}$ transforms as a *doublet* of SU(2) or a *triplet* of SU(3), for example:

$$\boldsymbol{\phi} := \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \longmapsto \exp\left\{\frac{\mathrm{i}}{2}\alpha^a\sigma_a\right\}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}. \tag{3.2.4}$$

One can verify this transformation leaves (3.1.16) invariant knowing that an action $U(\boldsymbol{\alpha}) \in \mathrm{SU}(2)$ satisfies $U^\dagger U = \mathbb{1}_2$. The Lagrangian for this doublet exhibits an internal SU(2) symmetry.

We can also consider the case for the Lagrangian for a free Dirac field (3.1.18), following [48, p. 8]. If we write Ψ as the sum of its Weyl spinors ψ_L, ψ_R using relevant projection operators (2.5.25), then:

$$\mathscr{L} = \overline{\psi}_L \mathrm{i}\gamma^\mu\partial_\mu\psi_L + \overline{\psi}_R \mathrm{i}\gamma^\mu\partial_\mu\psi_R - m^2(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R). \tag{3.2.5}$$

We noted at the end of the last chapter that our spinor theories were currently massless, so in the case $m = 0$ there is a clear U(1) symmetry for each Weyl spinor:

$$\begin{bmatrix}\psi_L\\ \psi_R\end{bmatrix} \longmapsto \begin{bmatrix}\mathrm{e}^{\mathrm{i}\alpha_L}\psi_L\\ \mathrm{e}^{\mathrm{i}\alpha_R}\psi_R\end{bmatrix}. \tag{3.2.6}$$

This would be a $\mathrm{U}_L \times \mathrm{U}_R$ internal symmetry. However, if we were to describe a massive spinor field then $m \neq 0$ and there exist mixing terms in the Lagrangian (3.2.5) that break this symmetry. In this case we collapse to a *single* U(1) symmetry if and only if $\alpha_L = \alpha_R$. This is our first example of symmetry breaking and why it is required to give certain fields their masses.

[7] Sidney Coleman, 1937 - 2007, American & Jeffrey Mandula, 1941 - , American.

[8] In reference to a phenomenon known as the **Aharonov-Bohm solenoid effect**, where in experiment a charged particle may experience a phase shift [53, p. 486] that actually *is* observable [56].

4
Spontaneous Symmetry Breaking

After carefully making our way through the background, algebraic formulations and symmetries, we now aim to understand how it all links together in the very specific cases. We discussed spin and how spin classifies the fundamental particles around us. In order to understand the Standard Model we will have to walk through it spin by spin and discuss how each class of particle can arise from studying various fields, and just as importantly describe their interactions with one another.

We are going to begin with particles exhibiting zero spin: **scalar bosons**. There exists only one of these in the Standard Model so one might wonder why there is such a long chapter describing it. The truth is it's complicated. First, we have to take the symmetries we have learnt to love and break them since the universe relies on broken symmetries just as much as unbroken, regular symmetries. Section 4.1 takes us through some examples of this occurrence for global group actions and uncovers Goldstone bosons. Section 4.2 generalises the idea of finding these bosons and how they must come about no matter the group in question. Finally, in Section 4.3 we discover what happens when we try to link symmetry breaking with the gauge theories discussed in the previous chapter. This directs us towards the **Higgs mechanism**, subsequently creating the spin-zero particle we are looking for.

4.1 Global breaking of symmetries

As noted previously, Lagrangians should here be thought of as the collection of objects that are symmetric under transformations from certain groups. Broken symmetries arise when Lagrangians contain a term that will later down the line violate the symmetry it was built on, often when considering the ground state of the theory [52, p. 18-19]. We have seen charged fields in the context of U(1) so far, but to motivate symmetry breaking we first consider a discrete symmetry before then finding a connection back to U(1) via a continuous symmetry. This presentation was inspired by Sections 2.1 & 2.2 of [49] and Sections 5.2 & 5.3 of [45].

4.1.1 Discrete symmetry breaking

We consider a real scalar field that possesses a discrete $\mathbb{Z}_2$ symmetry. The Lagrangian in question can be taken directly from a quantum field theory of a scalar field that interacts with itself, namely

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4, \tag{4.1.1}$$

by which the parity exchange $\phi \leftrightarrow -\phi$ leaves the Lagrangian invariant.

Note how this is different from standard free field theory. The free theory has exact solutions, but is considered elementary because particles do indeed interact with each other and it should be accounted for. To combat this, one adds the interacting term $\sim \lambda\phi^4$ to the potential. This is interpreted as ϕ interacting with itself with a **coupling constant** λ controlling the strength of this interaction. The theory is now completely unsolvable (see, eg, Chapter 4 of [23]), but that shouldn't deter us because we can analyse the potential of the theory and gain valuable information instead.

The potential of this theory is $V(\phi) = m^2\phi^2/2 + \lambda\phi^4/4$. If $\lambda = 0$ it is clear we turn back to the free theory and observe the simple harmonic potential as seen in Figure 4.1. A bounded potential requires $m^2 > 0$.

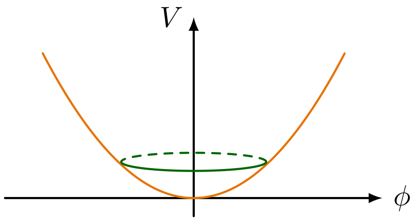

FIGURE 4.1: The simple harmonic potential.

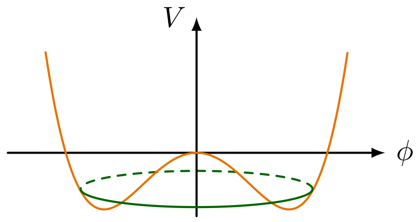

FIGURE 4.2: The Mexican hat potential.

With no energy, a particle would sit at the bottom of this potential well. An excitation in this field is what one would denote a particle, which would then begin internally spinning in quantised energy levels. This is pictorially shown above in green. The state of lowest energy would be the zero point of this field, the origin, by which the angular momentum operator attains no value. Since there is only one vacuum state at $\phi = 0$, it is clearly invariant under parity exchange and there has been no symmetry violations. In the case of the simple harmonic potential one notes that the only way for angular momentum to attain a value is if a particle "moves up the side of the wall" [91]: the field under this potential can only take new values by becoming excited.

In the interacting picture, by which $\lambda \neq 0$, it is required that $\lambda > 0$ for a bounded potential. It is not difficult to see that for any positive m^2 the new potential has the same shape as the simple harmonic potential (albeit steeper at the walls). The interesting physics arises when one considers $m^2 < 0$. A new potential arises: the Mexican hat potential, aptly named for its 3D representation. This is shown in Figure 4.2. The vacuum state is no longer at the origin and is instead replaced by two states of lowest energy values for which

$$\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = m^2\phi + \lambda\phi^3 = 0 \iff \phi_{\text{min}} = \pm\sqrt{-\frac{m^2}{\lambda}}. \tag{4.1.2}$$

In free theory, the ground state is the vacuum and the vacuum expectation value (VEV) is zero. In the Mexican hat picture, the VEV is nonzero and can take two values. Without loss of generality, define the VEV using the positive value. All analysis can be done with either ground state and this has been chosen for simplicity.

$$\nu := \langle\phi\rangle_0 = +\sqrt{-\frac{m^2}{\lambda}}. \tag{4.1.3}$$

It is now apparent that the vacuum does not exhibit the same $\mathbb{Z}_2$ symmetry that the Lagrangian does: the symmetry has spontaneously been broken. Performing a parity exchange allows the particle to jump between the two ground states. Although this seems to be a non-physical transformation, it serves as an example and the sentiment can be extrapolated to other, more physically-related symmetry groups.

To study further, one realises the potential can be written in terms of the VEV as

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - \nu^2)^2 + \text{constant}, \tag{4.1.4}$$

and the constant term can be removed; we are only concerned with the dynamics provided by the potential. As verification,

$$(\phi^2 - \nu^2)^2 = \phi^4 - 2\nu^2\phi^2 + \nu^4 = \phi^4 + \frac{2m^2}{\lambda}\phi^2 + \frac{m^4}{\lambda^2}, \tag{4.1.5}$$

using the defined VEV. Suppose we are now at this chosen ground state ν and we attempt to create a particle by making perturbations around this point: $\phi = \nu + \sigma$. The potential now reads off as

$$V(\sigma) = \frac{\lambda}{4}\left((\nu+\sigma)^2 - \nu^2\right)^2 = \frac{\lambda}{4}(\sigma^2 + 2\nu\sigma)^2 = \lambda\left(\frac{1}{4}\sigma^4 + \nu^2\sigma^2 + \nu\sigma^3\right). \tag{4.1.6}$$

There are two key things here to discuss. First, this potential is no longer parity-symmetric! While the full potential keeps this symmetry, first excitations around the minima do not. Tong states that it is in this sense in which the $\mathbb{Z}_2$ symmetry is 'hidden' or 'broken'; its purpose is to break and generate multiple ground states. Second, this perturbation allows us to read off the mass of the bare states particle as

$$m_\sigma^2 = 2\lambda\nu^2 = -2m^2 > 0. \tag{4.1.7}$$

It is imperative to note the difference between the physical-mass squared and the parameter-mass squared as in the potential. That is, m_σ^2 vs m^2. Within the field we are trying to describe, the particles created via our perturbation techniques have a mass different to that which we usually interpret in our Lagrangian (4.1.1). In some sense the mass of a particle this field creates is 'hidden', just like the symmetry the Lagrangian was built on. We can conclude that the mass of a particle can only be read off from the Lagrangian/potential if the vacuum state corresponds to the origin; the VEV is zero.

4.1.2 Continuous symmetry breaking

The aforementioned potentials can be extended into a more familiar theory by making the charged scalar field complex, giving us the Lagrangian

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi - \frac{1}{2}m^2\phi^\dagger\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2, \tag{4.1.8}$$

which is manifestly U(1)-invariant via phase shifts $e^{i\alpha}$. A near-equivalent phenomenon occurs: when the potentials are mapped out, the true Mexican hat is shown when we set the parameter $m^2 < 0$ and it is clear that the ground state will be made up of values satisfying

$$\nu^2 = -\frac{m^2}{\lambda}. \tag{4.1.9}$$

The set of ground states is a solution to this equation for ν and parametrises what is regarded as a **ground state manifold**, denoted $\mathcal{M}_0$. We build ground state manifolds by finding all possible field values where the potential is at a minimum, and is indeed a manifold because we are dealing with continuous transformations (Lie groups) and subgroups of Lie groups are themselves Lie subgroups [77]. In our complex space, with the case of a U(1) symmetry, the ground state manifold is simply a circle:

$$\mathcal{M}_0 := \{\nu : V(\nu) = V_{\text{min}}\} = \mathbf{S}^1. \tag{4.1.10}$$

Each of these states can be transformed into the other via an element in U(1). This means the ground state is not unique, and therefore a symmetry of the Lagrangian has been spontaneously broken in the vacuum. This is a more physical transformation than the parity exchange we saw in Section 4.1.1. As we did before in the discrete case, we can rewrite the potential of the complex scalar field Lagrangian (up to a constant term) as

$$V(|\phi|) = \frac{1}{4}\lambda(|\phi|^2 - \nu^2)^2. \tag{4.1.11}$$

The constant term can be neglected from analysis because the potential is used to describe how a field/particle *moves* in space. This involves gradients, which automatically eradicates the constants. This allows us to uncover what happens when we choose any one ground state as our VEV and perturb around it. Since this has become a discussion on circular motion, we can decompose the field into a radial part and phase parameter part:

$$\phi(x) = \rho(x)\mathrm{e}^{\mathrm{i}\vartheta(x)}. \tag{4.1.12}$$

Notice that variations in $\vartheta(x)$ will take us around the manifold $\mathcal{M}_0$ and variations in $\rho(x)$ will take us up and down the potential walls. This is important to realise because with an extra dimension comes an extra degree of freedom. This new formalism provides us with an extra, formally hidden part of the Lagrangian:

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\rho(x)\partial^\mu\rho(x) - \frac{1}{2}\rho^2\partial_\mu\vartheta\partial^\mu\vartheta - \frac{1}{4}\lambda(\rho^2 - \nu^2)^2. \tag{4.1.13}$$

Our analysis can now begin. The ground states are radially symmetric and lie wherever $\rho^2 = \nu^2$, but we already knew that. Note we can take excitations (that are really just radial oscillations) around a specific ground state. For our specific case this represents choosing a point on $\mathcal{M}_0$, a circle, and for simplicity we can choose the angular variable of the ground state to be zero so that the radial oscillations are completely real:

$$\rho(x) = \nu + \sigma(x). \tag{4.1.14}$$

Since ν is now a constant, the perturbed Lagrangian is

$$\begin{aligned}\mathscr{L} &= -\frac{1}{2}\partial_\mu\left(\nu + \sigma(x)\right)\partial^\mu\left(\nu + \sigma(x)\right) - \frac{1}{2}\left(\nu + \sigma(x)\right)^2\partial_\mu\vartheta\partial^\mu\vartheta - \frac{1}{4}\lambda\left(\left(\nu + \sigma(x)\right)^2 - \nu^2\right)^2 \\ &= -\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\left(\nu + \sigma\right)^2\partial_\mu\vartheta\partial^\mu\vartheta - \frac{1}{4}\lambda\left(\sigma^2 + 2\nu\sigma\right)^2.\end{aligned} \tag{4.1.15}$$

The interpreted mass of the excitation can be easily read off from the σ^2 coefficient, that being

$$m_\sigma^2 = 2\lambda\nu^2 = -2m^2 > 0, \tag{4.1.16}$$

just like the discrete case. Note here that again the mass of the excitation is not how it appears when first interpreted in the Lagrangian (4.1.8). Again this is nothing but a consequence of the fact the vacuum state of the potential in question is not unique and more importantly nonzero.

4.1.3 Faster than lightspeed—not!

At the end of all this, it can be unsettling to forget about the bigger picture and zoom into the fact that $m^2 < 0$ was suggested without prior reason. If one takes an interacting quantum field theory at face value, it looks like we are describing interacting particles with masses m^2. So what would it actually mean to use the existing theory and analyse the situation of negative mass?

There exist particles in theoretical physics called **tachyons** exhibiting negative mass. A consequence of this is that they would travel faster than lightspeed, breaking a key concept of special relativity. However, the particles we are studying do not travel faster than lightspeed. Instead we must change our perspective on what the 'mass' parameter is telling us. In potentials with $m^2 > 0$ we noted the origin was the ground state, and a stable one at that. Potentials shaped like the Mexican hat alternatively have the origin at an unstable maximum. It is deemed unstable because as we have demonstrated, any radial oscillations will immediately send excitations into the rim of the hat; from an unstable to a stable state.

Another way of thinking about our above analysis is taking an unstable stationary state and finding a new and improved stable one. By writing the field as some radial oscillation of its ground state, we uncovered the potential

$$V(\sigma) = \lambda\left(\nu^2\sigma^2 + \frac{1}{2}\nu\sigma^3 + \frac{1}{4}\sigma^4\right), \tag{4.1.17}$$

by which we determine the physical mass of the σ particle as $-2m^2 > 0$.

► This can happen in other areas of physics too; tachyons are rather important in the development of supersymmetry as they arise from the ground state of bosonic strings [32]. The vacuum of this theory is clearly unstable and thus to bring back stability the tachyon must be eliminated. One way in which to resolve this comes from supersymmetry, in which the reader is directed to Joe's Little Book of String [44]. Supersymmetry can be an exotic kind of symmetry that links bosons with fermions, but is not a symmetry we will explore in this report.

4.2 Fields in higher dimensions

Looking back at the continuous symmetries, nothing out of the ordinary occurred when taking the above results at face-value. The continuous symmetry was a generalisation of the discrete symmetry, and so of course the exact same mass had been revealed. The crucial difference between the two models, however, is noting that there was an extra degree of freedom we could play with: the radial mode $\rho(x)$ had its phase partner $\vartheta(x)$. Could we have said anything about its mass spectrum?

By looking at the Lagrangian (4.1.15) is it evident that our new degree of freedom $\vartheta(x)$ exists only in its gradient form. More specifically, there is no ϑ^2 term. This has the interpretation of being a massless boson field. We only encountered this field by breaking a continuous symmetry. There is an underlying theory here; it is a direct result of **Goldstone's**[1] **theorem**. Before we move to the theorem, it will be beneficial to highlight one last example in higher-dimensional space.

4.2.1 O(N) sigma models

When presenting theories about the strong force in 1960, Physicists Gell-Mann and Levy[2] discussed a 'σ model' that helped pair known Lagrangians with axial vector currents. Their paper highlighted a specific case of spontaneous symmetry breaking whereby a particular gauge transformation on O(4) left an O(3) subgroup unchanged, and in that sense the O(4) symmetry was broken [63].

The connection between the special unitary groups we are interested with and the orthogonal groups is different for each pairing, but recall that SU(2) is the double-cover of SO(3) and in fact U(1) $\cong$ SO(2). For the general picture we will follow [52] and keep in mind that there is a unitary connection for each orthogonal group. The reason for this is that the Lagrangian formalism for SU(N)-symmetric theories is slightly more complicated, and we will touch on them in Section 5.1. This formalism steers away from complex fields and allows us to focus on the higher-dimensional part of the description. That all being said, the sigma models drive us towards the regular orthogonal groups as opposed to the special ones, ignoring the disconnected component structure.

We begin with an N-component real field $\boldsymbol{\phi}$ whose Lagrangian is given by

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i\phi_i - \frac{1}{4}\lambda\left(\phi_i\phi_i\right)^2, \tag{4.2.1}$$

which is manifestly invariant under SO(N) transformations due to this group preserving the inner product, a result known from Linear Algebra. As we have seen a number of times the minimum from the Mexican hat potential produces a ring of minima that satisfy the equation $\phi_i\phi_i = -m^2/\lambda$ whenever $m^2 < 0$, for which the solutions we will (unsurprisingly) call $\boldsymbol{\nu}$.

For the purposes of analysis we need to choose one of these ground states, and a similar argument falls out. One can always align their coordinates such that the N^{th} component of $\boldsymbol{\phi}$ is the one to develop

[1] Jeffrey Goldstone, 1933 - , British.

[2] Murray Gell-Mann, 1929 - 2019, American, Nobel Prize 1969 & Maurice Lévy, 1922 - 2022, French.

the nonzero VEV:

$$\boldsymbol{\nu} := \langle \boldsymbol{\phi} \rangle_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \nu \end{bmatrix}. \tag{4.2.2}$$

We can play a quick counting game. Due to antisymmetry the orthogonal symmetry groups have $N(N-1)/2$ independent generators we call T_{ij}. However, by virtue of our coordinate choices, there is clearly an invariant subspace $\mathrm{SO}(N-1)$ that did in fact reach a zero VEV— this is evident by looking at the first $N-1$ components of $\langle \boldsymbol{\phi} \rangle_0$.

$\mathrm{SO}(N-1)$ has $(N-1)(N-2)/2$ independent generators we denote by t_{ij}. Let the remaining elements of the broken symmetry be denoted S_i. The reason for giving these elements names is to realise

$$t_{ij} = T_{ij} \quad \text{for} \quad i,j \neq N \quad \text{and} \quad S_i = T_{iN}. \tag{4.2.3}$$

This also tells us there are $N-1$ independent S_i:

$$\frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1. \tag{4.2.4}$$

The natural progression of this technique would be to parametrise the N fields via their one-dimensional 'radial' part and $(N-1)$-dimensional 'angular' parts and use the standard perturbation technique around the ground state:

$$\boldsymbol{\phi}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\vartheta_i S_i} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \nu + \sigma(\mathbf{x}) \end{bmatrix}, \tag{4.2.5}$$

whereby we have parametrised the vacuum manifold $\mathcal{M}_0$ via ϑ_i and created an excitation $\sigma(\mathbf{x})$, and i of course runs from $1 \leqslant i \leqslant N-1$.

We can utilise the algebra of $\mathrm{SO}(N)$ (2.1.11) and note that the generators T_{ij} (and therefore S_i) have the structure

$$(T_{ij})_{kl} = -\mathrm{i}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \implies (S_i)_{kl} = -\mathrm{i}(\delta_{ik}\delta_{Nl} - \delta_{il}\delta_{Nk}). \tag{4.2.6}$$

This allows us to see precisely how the above angular transformations act on the radial perturbation $(\nu+\sigma)\delta_{jN}$. To first order, the j^{th} component of the perturbation is

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\vartheta_i S_i}(\nu+\sigma)\delta_{jN} &\approx (\nu+\sigma)(1 + \mathrm{i}\vartheta_i S_i + \dots)\delta_{jN} \\ &= (\nu+\sigma)\delta_{jN} + \mathrm{i}(\nu+\sigma)\vartheta_i(S_i)_{jN} + \mathcal{O}(\boldsymbol{\vartheta}^2). \end{aligned} \tag{4.2.7}$$

The first term is the unperturbed object; S_i then act to perturb the rest of the state as such:

$$\begin{aligned} (S_i)_{jN} &= -\mathrm{i}(\delta_{ij}\delta_{NN} - \delta_{iN}\delta_{Nj}) \\ &= -\mathrm{i}\delta_{ij} \quad \text{because} \quad i \neq N \\ \implies \boldsymbol{\phi}(\mathbf{x})_j &= (\nu+\sigma)\delta_{jN} + (\nu+\sigma)\vartheta_j, \end{aligned} \tag{4.2.8}$$

keeping in mind that $\boldsymbol{\vartheta}$ is $(N-1)$-dimensional and so $\vartheta_N = 0$.

What exactly is this saying? The role of S_i is to perform a multi-dimensional phase transition on the ground state. One notes that only the $(N-1)$-dimensional subspace is affected. So, once the fields are multiplied together in the Lagrangian (4.2.1), the 'angular' variations in the quadratic part cancel out and there is no sign of $\vartheta_i\vartheta_i$. This results in

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\partial_\mu\vartheta_i\partial^\mu\vartheta_i - \frac{1}{2}m^2(\nu+\sigma)^2 - \frac{1}{4}\lambda(\nu+\sigma)^4, \tag{4.2.9}$$

ignoring some of the higher-order interaction terms. The bare excitation $\sigma(\mathbf{x})$ has an interpreted mass of $-2m^2$ (as per all of the previous cases) and the $N-1$ ϑ_i fields are massless. Each generator S_i breaking a symmetry of the Lagrangian corresponds to a massless field, sometimes referred to as a massless **mode**.

4.2.2 Goldstone's theorem

Could it be a coincidence that the number of massless modes is the same as the number of broken generators? It seems a sensible question, after all we have only discussed groups closely linked with N-spherical rotations; surely this fact comes by virtue of the representation of $\mathrm{SO}(N)$?

In fact, it was the symmetry groups of the potential we were wrongfully zooming in on; the theory should be, and will be, extended to general groups and general potentials. We again follow a structure similar to [52]. Take once more a Lagrangian density of N real scalar fields ϕ_i but now with some arbitrary potential:

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} - V(\boldsymbol{\phi}), \tag{4.2.10}$$

where $V(\boldsymbol{\phi})$ is some polynomial in $\boldsymbol{\phi}$ that is invariant under transformations $\boldsymbol{L}$ in some Lie group G. For dimensionalities to be consistent we must have that G has N independent generators T_a (so its Lie algebra $\mathfrak{g}$ has basis $\{\mathrm{i}T_a\}$), and for physicality $\boldsymbol{\phi}$ must transform under an N-dimensional matrix representation of G. One is then inclined to write transformations as $\boldsymbol{L} = \exp\{\mathrm{i}\vartheta^a T_a\}$, defining ϑ^a to parametrise the vacuum manifold $\mathcal{M}_0$. Variations in the field are then linear transformations of the field itself:

$$\delta\boldsymbol{\phi} = \mathrm{i}\vartheta^a T_a\boldsymbol{\phi}. \tag{4.2.11}$$

The potential V is invariant under G, so by the chain rule one can write

$$0 = \delta V = \frac{\partial V}{\partial\phi_i}\delta\phi_i = \mathrm{i}\frac{\partial V}{\partial\phi_i}(\vartheta^a T_a)_{ij}\phi_j. \tag{4.2.12}$$

The vacuum manifold parameters ϑ^a are, of course, arbitrary and the analysis should not be affected by change of higher-dimensional 'phase'. Most importantly they are probably not zero, which means it safe to ensue the following N equations to solve for ν:

$$0 = \frac{\partial V}{\partial\phi_i}(T_a)_{ij}\phi_j, \tag{4.2.13}$$

for all values of a. Now, consider the second derivative of the potential with respect to a second field component ϕ_k and note its result:

$$0 = \frac{\partial}{\partial\phi_k}\left(\frac{\partial V}{\partial\phi_i}(T_a)_{ij}\phi_j\right) = \frac{\partial^2 V}{\partial\phi_i\partial\phi_k}(T_a)_{ij}\phi_j + \frac{\partial V}{\partial\phi_i}(T_a)_{ij}\delta_{jk}. \tag{4.2.14}$$

At the ground state $\boldsymbol{\phi} = \boldsymbol{\nu}$ the first derivative is zero, giving us

$$0 = \left.\frac{\partial V}{\partial\phi_i}\right|_{\boldsymbol{\phi}=\boldsymbol{\nu}} \implies 0 = \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_k}\right|_{\boldsymbol{\phi}=\boldsymbol{\nu}}(T_a)_{ij}\nu_j. \tag{4.2.15}$$

Taking the second derivative might have seemed ad hoc, but we can see exactly where it arises naturally when doing perturbative analysis as usual. With $\boldsymbol{\phi}(\mathbf{x}) = \boldsymbol{\nu} + \boldsymbol{\sigma}(\mathbf{x})$, the potential of the excited state

$\boldsymbol{\sigma}(\mathbf{x})$ can be expanded to reveal

$$V(\boldsymbol{\sigma}) = V(\boldsymbol{\phi} - \boldsymbol{\nu}) \approx V(\boldsymbol{\nu}) + \frac{\partial V}{\partial \phi_i}(\boldsymbol{\phi} - \boldsymbol{\nu})_i + \frac{1}{2}\frac{\partial^2 V}{\partial \phi_i \partial \phi_k}(\boldsymbol{\phi} - \boldsymbol{\nu})_i(\boldsymbol{\phi} - \boldsymbol{\nu})_k + \mathcal{O}(\boldsymbol{\sigma}^3), \tag{4.2.16}$$

with all derivatives evaluated at the ground state. With the results from (4.2.13), this gets rid of the linear term and leaves only the second derivative term. Recall we have complete freedom over the choice of coordinates, so in fact we can get rid of the constant term too just by shifting our reference point.

Note how the second derivative of V appears in a term which can only be described as kinetic energy of the bare state $\boldsymbol{\sigma}$ - we have the $1/2$ and the field squared, so the only thing left to do is determine what 'physical mass' should be. We hence define the **mass matrix** M_{ij}^2 to be this second derivative term so that

$$M_{ij}^2 (T_a)_{jk}\nu_k = 0 \quad \text{for each } a \in 1, \dots, \dim(\mathrm{G}). \tag{4.2.17}$$

Physical masses of the $\boldsymbol{\sigma}$ fields are eigenvalues of this mass matrix. We distinguish the cases of the type of eigenvalues this matrix produces with extra notes from Section 5.4 of [45]. Let H be the M-dimensional subgroup of G that does not break the symmetry and leaves the ground state symmetric:

$$\boldsymbol{L}\langle\boldsymbol{\phi}\rangle_0 = \langle\boldsymbol{\phi}\rangle_0 \quad \text{for} \quad \boldsymbol{L} \in \mathrm{H}\,. \tag{4.2.18}$$

If $T_a \in \mathfrak{g}$ also happens to be a generator of H then it would act on $\langle\boldsymbol{\phi}\rangle_0$ just like S_i in the previous section (4.2.8); to first order the in the expansion of the field variation (4.2.11) there is no perturbation and so

$$(T_a)_{jk}\nu_k = 0, \tag{4.2.19}$$

meaning (4.2.17) becomes trivial and we cannot extract any useful information about the mass matrix M_{ij}^2. In contrast, there must now be a subspace G / H that breaks the symmetry. This spontaneous symmetry breaking creates a vacuum manifold $\mathcal{M}_0$ with dimension dim(G) − dim(H), which here is $N - M$.

The linear transformation is no longer trivial and so there are $N - M$ vectors $(T_a)_{jk}\nu_k$ that are nonzero. (4.2.17) can thus be treated as an eigenvalue problem where all of the eigenvalues are zero. Subsequently, the fields described by this subspace G / H must be massless: $\{T_a\,\boldsymbol{\nu}\}$ span this $(N-M)$-dimensional space, and as a consequence there must exist $N - M$ massless modes in the theory.

Goldstone's theorem

Each manifestation of a broken continuous global symmetry brings about a nonzero VEV of a massless boson field. If the symmetry group of a Lagrangian breaks from G to H, there exist exactly

$$\dim(\mathrm{G}) - \dim(\mathrm{H}) = \dim(\mathcal{M}_0) \tag{4.2.20}$$

massless modes. These are crowned the **Goldstone bosons** [66].

This was a result presented in 1962 in a journal piece by Goldstone himself alongside Salam and Weinberg[3].

4.3 The Higgs mechanism

At this point in particle physics, the early 1960s, a lot of mathematical theory had been proposed without much physical evidence [5]. In fact, when Goldstone's theorem was proved in 1962 there was no claim that these Goldstone bosons existed; only they must appear as a result of a spontaneously

[3] Adbus Salam, 1926 - 1996, Pakistani & Stephen Weinberg, 1933 - 2021, American, both Nobel Prize 1979.

broken continuous symmetry. One might ask why they appear from the maths when the experiments show otherwise. If we say a symmetry is allowed to be broken yet an unexpected result arises, then our theory must be tweaked.

The missing piece in this puzzle is credited to Higgs[4]. So far physicists had only discussed spontaneous breaking of global symmetries - but only few had dared to figure out what happens when the symmetry is *gauged*.

4.3.1 Gauge invariance

In studying past courses we know that at the heart of field theories for particle physics lies gauge invariance: the idea that "nature is best described using a descriptively redundant language" [36, Abstract]. The idea stems from demanding that physics is invariant when the group transformations depend on our exact position in spacetime. We take inspiration from [51].

The motivation for such theories came from Maxwell's[5] theory of electromagnetism. Historically, the use of a 4-vector potential was used as a conveniant way to solve the differential equations—but we now know that electromagnetism can be constructed by demanding our theory has an internal U(1) symmetry, with the extra condition that the complex phase of rotation is parametrised, now with electric charge e included: $\exp\{\mathrm{i}e\alpha\} \sim \exp\{\mathrm{i}e\alpha(x^\mu)\}$. If the Lagrangian for a charged scalar field (**??**) is to still be invariant, an entirely new field must be introduced. A familiar calculation yields:

$$\underbrace{\partial_\mu\left(\mathrm{e}^{\mathrm{i}e\alpha}\phi\right) = \mathrm{e}^{\mathrm{i}e\alpha}\partial_\mu\phi}_{\text{global U(1)}} \iff \underbrace{(\partial_\mu - \mathrm{i}eA_\mu)\left(\mathrm{e}^{\mathrm{i}e\alpha(x)}\phi\right) = \mathrm{e}^{\mathrm{i}e\alpha(x)}\left(\partial_\mu - \mathrm{i}eA_\mu\right)\phi,}_{\text{local U(1)}} \tag{4.3.1}$$

for some *gauge field* A_μ that obeys $\delta A_\mu = \partial_\mu\alpha(x)$. In the context of a pure U(1)-invariant theory, we can refer to this as the photon field[6]; it is a vector field and therefore must describe a spin-1 particle, a boson, a force carrier. The gradient of the charged scalar field, $\partial_\mu\phi$, may only appear in the Lagrangian in conjunction with the photon field. For this reason we choose to compactify notation and use the **covariant derivative** $\mathcal{D}_\mu := \partial_\mu - \mathrm{i}eA_\mu$ in place of this combination.

Charged fields must therefore couple to the photon field with coupling strength e—this is **minimal coupling**. To fully appreciate the theory of electromagnetism we must introduce the kinetics of the photon field, which can be compactified using the field strength tensor $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. This is clearly a U(1)-invariant quantity since partial derivatives commute,

$$\delta F_{\mu\nu} = \partial_\mu\partial_\nu\alpha(x) - \partial_\nu\partial_\mu\alpha(x) = 0, \tag{4.3.2}$$

and so as long as a Lagrangian is constructed from this tensor it will be manifestly phase invariant.

Scalar electrodynamics

We present the Lagrangian for **scalar electrodynamics**:

$$\mathscr{L}_{\mathrm{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\mathcal{D}_\mu\phi)^\dagger\mathcal{D}^\mu\phi - \frac{1}{2}m^2\phi^\dagger\phi. \tag{4.3.3}$$

The constant prefactors there to ensure the Euler-Lagrange equations of motion result in Maxwell's equations.

[4] Peter Higgs, 1929-2024, British, Nobel Prize 2013.
[5] James Clerk Maxwell, 1831 - 1879, Scottish.
[6] We will see in Section 5.2 that the real photon field arises as a separate U(1) subgroup of $\mathrm{U}(1)_Y \times \mathrm{SU}(2)$.

► Interestingly, all four Maxwell equations can be written in one single equation using Clifford algebras:

$$(\partial_0 + \boldsymbol{\nabla})F = \rho - \mathbf{J} \text{ with APS [38, p. 14-16, 25–26]}; \quad \boldsymbol{\nabla} F = J \text{ with STA [39, p. 3-5, 25]}. \tag{4.3.4}$$

Of course, the notation is hiding a lot of information, but it is once again useful to see that Clifford algebras make appearances in other notable areas of physics.

One might like to relate this for the Lagrangian for a vector field we wrote down above (3.1.17). If this is to be the case, then we would be missing a term proportional to $m^2 A_\mu A^\mu$, but this term violates gauge invariance:

$$A_\mu A^\mu \longmapsto \big(A_\mu + \partial_\mu \alpha(x)\big)\big(A^\mu + \partial^\mu \alpha(x)\big) = A_\mu A^\mu + 2A_\mu \partial^\mu \alpha(x) + \big(\partial_\mu \alpha(x)\big)^2. \tag{4.3.5}$$

A term like this therefore cannot appear in our Lagrangian, so to ensure this we must set $m^2 = 0$; the photon must not have a mass.

This is not just reflected in electromagnetism, however. A less-trivial example comes when we consider a scalar field $\boldsymbol{\phi}$ that transforms as an SU(2) doublet as in (3.2.4). A gauge transformation would require $\alpha^a = \alpha^a(x^\mu)$ for $a = 1, 2, 3$. We will discuss this in more detail in Section 5.2 when we consider electroweak interactions, but the story remains the same. This gauge transformation would require a covariant derivative of the form

$$\mathcal{D}_\mu = \partial_\mu - \mathrm{i}e\boldsymbol{A}_\mu = \partial_\mu - \mathrm{i}e{A^a}_\mu \sigma_a, \tag{4.3.6}$$

for three gauge fields ${A^1}_\mu, {A^2}_\mu, {A^3}_\mu$ each obeying $\delta {A^a}_\mu = \partial_\mu \alpha^a(x) + \varepsilon^{abc}\alpha^b {A^c}_\mu$.

► This definition of an SU(2) gauge transformation for ${A^a}_\mu$ has been seen in previous courses, but we will treat it more carefully when reviewing the SU(2) algebra in the indicated section.

Mass terms for the three fields would be proportional to $M^2 \boldsymbol{A}_\mu \boldsymbol{A}^\mu$ for a matrix of mass values M^2. Again, this object violates local gauge invariance with the transformations we have introduced, and so the term cannot appear in the Lagrangian for a theory with gauge group SU(2).

As before, this requires setting $M^2 \equiv \boldsymbol{0}$, and hence the vector boson in this theory is manifestly massless. But this feature is completely independent of the gauge group of the theory and/or vector boson field in question— any squared vector field may not belong inside a Lagrangian since it is not Lorentz invariant. This poses an issue because, as we have mentioned before, there exist spin-1 particles that exhibit mass, namely the weak bosons.

4.3.2 Gauge symmetry breaking and the Abelian Higgs model

This discussion was inspired by Section 5.5 of [45] with other references detailed along the way. We begin with the scalar electrodynamics Lagrangian (4.3.3) with the potential $V(\phi, \phi^\dagger)$ being replaced by the Mexican hat:

$$\mathscr{L}_{\mathrm{H}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\mathcal{D}_\mu\phi)^\dagger \mathcal{D}^\mu \phi - \frac{1}{4}\lambda\left(|\phi|^2 - \nu^2\right)^2, \tag{4.3.7}$$

Together with the field strength tensor and covariant derivative,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ and } \mathcal{D}_\mu \phi = \partial_\mu \phi - \mathrm{i}eA_\mu \phi, \tag{4.3.8}$$

this forms the **Abelian Higgs model** [69]. The mechanism itself begins with locating the VEV ν, defined by all points satisfying $|\phi|^2 = \nu^2$ as before. This was previously recognised as the vacuum manifold $\mathcal{M}_0 = \mathbf{S}^1$ parametrised by the phase ϑ of ϕ. This will now change, since we are under the gauge framework by which

$$\phi \mapsto \mathrm{e}^{\mathrm{i}e\alpha(x)}\phi \text{ and } A_\mu \mapsto A_\mu + \partial_\mu \alpha(x). \tag{4.3.9}$$

That is, the phase has been parametrised by $\alpha(x)$ and is now dependent on spacetime coordinates. This will affect the derivative terms in the Lagrangian when we perturb around the ground state.

► In the previous global cases, we were free to choose which ground state to perturb around without loss of generality. But we do not hold the same freedom of choice anymore: gauge symmetries were introduced so that any field configurations obeying this symmetry are to be interpreted as one and the same. Because all vacua are symmetric under a U(1) gauge transformation, there can be only one true ground state.

Suppose we have the ground state for the given phase $\phi = \mathrm{e}^{\mathrm{i}\vartheta(x)}\nu$ and perturb it slightly via a new field $\eta(x)$:

$$\phi(x) = \mathrm{e}^{\mathrm{i}\vartheta(x)}\left(\nu + \eta(x)\right), \quad \eta(x) \ll 1. \tag{4.3.10}$$

Since the change of phase now depends on spacetime coordinates, the derivative term in the Lagrangian comes with extra pieces:

$$\begin{aligned}
\mathcal{D}_\mu\phi &= \partial_\mu\big(\mathrm{e}^{\mathrm{i}\vartheta(x)}\left(\nu + \eta(x)\right)\big) - \mathrm{i}eA_\mu\big(\mathrm{e}^{\mathrm{i}\vartheta(x)}\left(\nu + \eta(x)\right)\big) \\
&= \mathrm{e}^{\mathrm{i}\vartheta(x)}\Big(\big(\partial_\mu + \mathrm{i}\partial_\mu\vartheta(x)\big)\big(\nu + \eta(x)\big) - \mathrm{i}eA_\mu\big(\nu + \eta(x)\big)\Big) \\
&= \mathrm{e}^{\mathrm{i}\vartheta(x)}\partial_\mu\eta(x) + \mathrm{i}\big(\partial_\mu\vartheta(x) - eA_\mu\big)\phi.
\end{aligned} \tag{4.3.11}$$

Similarly,

$$(\mathcal{D}_\mu\phi)^\dagger = \mathrm{e}^{-\mathrm{i}\vartheta(x)}\partial_\mu\eta(x) - \mathrm{i}\big(\partial_\mu\vartheta(x) - eA_\mu\big)\phi^\dagger. \tag{4.3.12}$$

The perturbed Lagrangian becomes

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial_\mu\eta\partial^\mu\eta - (\nu+\eta)^2\left(\partial_\mu\vartheta - eA_\mu\right)\left(\partial^\mu\vartheta - eA^\mu\right) - \frac{\lambda}{2}\eta^2(\eta + 2\nu)^2, \tag{4.3.13}$$

with the potential factored as in (4.1.15). This has immediately shown us that the interpreted mass of the η particle is the coefficient of η^2, $2\lambda\nu^2$, as expected. The gradient field for the Goldstone boson $\partial_\mu\vartheta$ is also kept. What is more interesting, though, is realising the coupling that is going on here: $\partial_\mu\vartheta$ only appears in conjunction with the field A_μ.

But this is a *gauge* field! One has complete freedom of how it transforms as part of a gauge symmetry. We can fix the gauge transformation so that

$$A'_\mu := A_\mu - \frac{1}{e}\partial_\mu\vartheta. \tag{4.3.14}$$

This has been chosen specifically so that under a U(1) gauge transformation, one observes

$$A'_\mu \mapsto A_\mu + \partial_\mu\alpha(x) - \frac{1}{e}\partial_\mu(\vartheta + e\alpha) = A_\mu - \frac{1}{e}\partial_\mu\vartheta = A'_\mu, \tag{4.3.15}$$

so A'_μ does not change! The trick was subtle: under this transformation, the phase ϑ of ϕ picks up an extra rotation of $e\alpha$, meaning

$$\vartheta(x) \mapsto \vartheta(x) + e\alpha(x). \tag{4.3.16}$$

By changing variables, we have fixed another gauge and effectively set $\vartheta(x) = 0$. This is **unitarity gauge** [83], and it completely eliminates the term involving the gradient of $\vartheta(x)$ in the above Lagrangian (4.3.13). Moreover, the field strength tensor $F_{\mu\nu}$ is not affected since

$$F_{\mu\nu} = \partial_\mu\left(A'_\nu + \frac{1}{e}\partial_\nu\vartheta\right) - \partial_\nu\left(A'_\mu + \frac{1}{e}\partial_\mu\vartheta\right) = \partial_\mu A'_\nu - \partial_\nu A'_\mu =: F'_{\mu\nu}, \tag{4.3.17}$$

and partial derivatives commute: the gradient of $\vartheta(x)$ perfectly cancels out. We are now left with

$$\mathscr{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \partial_\mu\eta\partial^\mu\eta - e^2(\nu+\eta)^2A'_\mu A'^\mu - \frac{\lambda}{2}\eta^2(\eta+2\nu)^2. \tag{4.3.18}$$

The proclaimed Goldstone boson is said to have been *gauged away*. We can understand this better by considering the degrees of freedom in this system. In the Abelian Higgs model (4.3.7) there were two degrees of freedom for the complex scalar ϕ and another two for the massless photon field A_μ; the two polarisations of light. But now in the perturbed Lagrangian there exists the massive scalar boson η and the vector field A'_μ which *must* have three degrees of freedom in order for the dimensionalities to line up. This third degree of freedom comes in the form of the Goldstone mode $\vartheta(x)$ which is absorbed by the once-massless photon—this is how one should interpret the switch to unitary gauge (4.3.14). We must now be describing a massive vector field that embeds a longitudinal polarisation.

This analysis offers a different interpretation of Goldstone modes and Goldstone's theorem when dealing with gauge theories. Since the modes no longer appear in our Lagrangian, they would not exist in nature and may be seen in literature as 'would-be' Goldstone bosons. The dimension of our ground state manifold now counts the number of massive gauge bosons when spontaneous symmetry breaking occurs, and we restrict ourselves to unitarity gauge. For the case of scalar electrodynamics this has been displayed above; the U(1) ground state manifold is a circle, a one-dimensional manifold, and this predicts the massive gauge boson A'_μ. All of the massless would-be Goldstone bosons have been completely absorbed to bring about massive fields.

The way in which a spontaneously broken gauge symmetry allows for the field A_μ to 'eat' the Goldstone boson is the **Higgs mechanism**, named in recognition of Higgs' contribution to the theoretical sciences. The massive scalar boson η discovered in the making was the first candidate for the **Higgs boson**: a spin-zero particle that belongs in the Standard Model whose purpose is to allow massive particles to be mathematically described in a gauge-symmetric way.

► As it often turns out in theoretical subjects, Higgs was not the only one to be discussing broken symmetries and masses of gauge bosons at this time. Credit should also be given to Brout and Englert, and Guralnik, Hagen and Kibble. In three separate teams, these six scientists formed the theory being discussed here [55, 68, 69]. Alas, only two of them received the Nobel Prize in Physics, and only one was blessed with the naming [1].

4.3.3 The photon mass: can we fix it?

Now, the eagle-eyed will notice a problem with the above Lagrangian, because of course it was too good to be true. A_μ was originally denoted the photon field, and we saw earlier (**??**) that the photon field couldn't attain a mass because there did not exist an $(A_\mu)^2$ term in the Lagrangian—it wouldn't be gauge invariant. But under the Higgs mechanism we have contrived an $(A_\mu)^2$ term and therefore given the photon a mass. What went wrong here?

In fact, nothing has gone wrong; there are two reasons for this. First, photons gaining mass is a relevant phenomena in superconductivity. That strays too far away from this report, though if this sparks interest the reader is directed to the relevant section in Tong's notes, Chapter 44 of Lancaster's book and a book dedicated to superconductivity [12, 20, 49]. Second, back to the land of gauge symmetry, it is nothing but our interpretation of what the purpose of the U(1) gauge symmetry actually represents in the context of the Standard Model, as opposed to electromagnetism as its own separate theory. This also explains why we proposed η as the 'first candidate' for the Higgs boson: its actual apparition comes in the context of electroweak theory.

The open-endedness of whether we can find a fixture serves as a perfect segue into the next chapter.

5 The Weak Force as a Gauge Theory

The mechanism we studied in the last chapter was groundbreaking in that it offered a solution to one of the main issues in the early days of particle physics, namely that massless Goldstone bosons were theorised and yet never observed [45, p. 93]. We uncovered that under the blanket of gauge theories we could interpret the modes as 'would-be' Goldstone bosons that massless fields absorb into their polarisations and in turn generate a mass. A new problem is now faced, however. During this procedure we accidentally gave the photon a mass, which is certainly nonphysical based on experimentation [11].

Thankfully, this chapter promises to offer a solution to the massive photon while also providing a consistent theory for another sector of the Standard Model. We cannot complete the Standard Model without the strong and weak nuclear forces. One might be inclined to suggest they also arise from gauge theories, perhaps the ones directly above U(1) in the hierarchy of special unitary groups: SU(2) and SU(3). What we should be able to do now is apply the Higgs mechanism to these gauge groups and discover what else can occur under spontaneous symmetry breaking. They will be more involved since all special unitary groups (aside from U(1)) are non-Abelian and thus must be treated with more care.

In Section 5.1 we discuss what gauge transformations look like when we consider general higher dimensions, leading us to Yang-Mills fields. This will make zooming into the specific algebras a lot easier. After another recap of the SU(2) algebra we cleverly combine it with our theory for U(1) and have a first look at how electromagnetism and the weak force are actually unified via **electroweak theory** in Section 5.2. It is historically a theory of fermions, but we will look at the bosonic case first before we dive into the full theory in Section 5.3.

5.1 Yang-Mills fields

The most important results from studying non-Abelian theory in previous courses were how each of our favourite characters transformed under a gauge transformation in $\mathrm{SU}(N)$ where $N \geqslant 2$. We follow arguments inspired by [49, p. 28-32].

Consider an N-component charged field $\phi^i \in \mathbb{C}^N$ (that may be scalar-valued or spinor-valued) transforming under some representation of a gauge group $\mathrm{SU}(N)$ with Lie algebra basis $\{\mathrm{i}t_a\}$ and coupling strength g. The number of basis elements is $\dim(\mathrm{SU}(N)) = N^2 - 1$, which are labelled by the *colour* index a. They obey the following Lie bracket and trace condition:

$$[t_a, t_b] = \mathrm{i} f_{abc} t_c \quad \text{and} \quad \mathrm{tr}(t_a t_b) = \frac{1}{2}\delta_{ab}, \tag{5.1.1}$$

introducing the **structure constants** f^{abc} of $\mathfrak{su}(N)$.

Definition	Transformation
Charged field ϕ^i	$g\phi^i$
$\mathcal{D}_\mu\phi^i$	$g\left(\mathcal{D}_\mu\phi^i\right)$
Covariant derivative $\mathcal{D}_\mu$	$g\left(\mathcal{D}_\mu\right)g^{-1}$
Gauge field $A^a{}_\mu$	$g\big(A^a{}_\mu + \frac{\mathrm{i}}{g}\partial_\mu\big)g^{-1}$
Field strength tensor $F^a{}_{\mu\nu}$	$g\left(F^a{}_{\mu\nu}\right)g^{-1}$

TABLE 5.1: Gauge transformations on objects by some element $g(x) \in \mathrm{SU}(N)$.

The following are useful to recall:

- $\boldsymbol{\phi}$ and $\mathcal{D}_\mu\boldsymbol{\phi}$ live in the **defining** representation of $\mathrm{SU}(N)$, where

$$\mathcal{D}_\mu\phi^i = \partial_\mu\phi^i - \mathrm{i}gA^a{}_\mu t_a\phi^i. \tag{5.1.2}$$

- $\mathcal{D}_\mu$, $\boldsymbol{A}_\mu$ and hence $\boldsymbol{F}_{\mu\nu}$ live in the **adjoint** representation of $\mathrm{SU}(N)$, where

$$\boldsymbol{F}_{\mu\nu} = F^a{}_{\mu\nu}t_a := \frac{1}{g}\mathrm{i}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu - gf^{abc}A^b{}_\mu A^c{}_\nu t_a, \tag{5.1.3}$$

  is the non-Abelian field strength tensor in a representation of $\mathrm{SU}(N)$.

The explicit gauge transformations are summarised in Table 5.1. Note the transformation of $A^a{}_\mu$ in particular is definitely in the adjoint representation, but it also carrises through its originally-defined gauge transformation that is familiar to us.

Now, to construct a Lagrangian in the context of a non-Abelian gauge theory, we must find all gauge invariant quantities. Previously this was done by squaring the field strength tensor by virtue of its definition, but that is no longer possible:

$$\boldsymbol{F}_{\mu\nu} \longmapsto g\left(\boldsymbol{F}_{\mu\nu}\right)g^{-1} \implies \boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu} \longmapsto g\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right)g^{-1}. \tag{5.1.4}$$

This is not a gauge-invariant quantity! What we must consider instead is its trace, which is gauge invariant due to its cyclic property:

$$\begin{aligned}\operatorname{tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) &\longmapsto \operatorname{tr}\left(g\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}g^{-1}\right)\\ &= \operatorname{tr}\left(g^{-1}g\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) = \operatorname{tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right).\end{aligned} \tag{5.1.5}$$

Thus, any quantity constructed via the trace of this tensor product will be automatically gauge invariant. This idea lead to the discovery of the **Yang-Mills**[1] **Lagrangian** [87]:

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{2g^2}\operatorname{tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) = -\frac{1}{4g^2}F^a{}_{\mu\nu}F_a{}^{\mu\nu}, \tag{5.1.6}$$

with the constant pre-factor again included to comply with existing Maxwell theory. We lose the factor of 2 in the trace version by virtue of how we defined the basis elements (5.1.1). Yang and Mills produced this Lagrangian in the context of isotopic spin in 1954, almost a decade before the Higgs mechanism began playing a role in particle physics. The Yang-Mills equations of motion can be

[1] Yang Chen-Ning, 1922 -, Chinese, Nobel Prize 1957 & Robert Mills, 1927-1999, American.

derived by minimising the action produced by this Lagrangian, giving

$$\mathcal{D}_\mu \boldsymbol{F}^{\mu\nu} = 0 \iff \partial_\mu \boldsymbol{F}^{\mu\nu} + g f_{abc} A^a{}_\mu F^{b\mu\nu} t_c = 0. \tag{5.1.7}$$

What is interesting to note about these equations of motion is that they contain lots of nonlinear terms in $A^a{}_\mu$. This provides evidence that Yang-Mills should be considered a theory that interacts with itself. This contrasts with Maxwell theory (that is itself a Yang-Mills theory but with gauge group U(1)) that is completely free and contains zero interacting terms. Moreover, this highlights the importance of the gauge field $\boldsymbol{A}_\mu$. When looking back at Maxwell theory, the equations of motion (and indeed the Lagrangian) could be written purely in terms of the fields $\mathbf{E}$ and $\mathbf{B}$. The 'photon' field A_μ was introduced purely to compactify the notation. But for Yang-Mills the gauge field $\boldsymbol{A}_\mu$ is *required* for the equations of motion—this is because it now transforms in a non-trivial way that involves commutators, which would vanish in an Abelian theory.

► An interesting note is that classically, the Yang-Mills fields are massless; the only source terms we have are from the field strength tensor. However, upon quantisation, there emerges a theory that predicts the interaction of massive particles [49, p. 32]. It is noted in this reference that the exact origins of this are not well-understood, and in fact the ability to construct an accepted quantum field theory with a mass gap would win us $1 million [97, 8].

In the case of a charged field, one studies a Lagrangian similar to

$$\mathscr{L} = -\frac{1}{2g^2}\operatorname{tr}\left(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right) - (\mathcal{D}_\mu\phi)^\dagger\mathcal{D}^\mu\phi - V(\phi,\phi^\dagger). \tag{5.1.8}$$

For the context of electroweak symmetry breaking it is not too hard to believe that the potential we are concerned with is the Mexican hat potential.

5.2 Electroweak symmetry breaking

A common misconception when first understanding the Standard Model via gauge groups is that the three gauge groups in question correspond to electromagnetism, the weak nuclear force and the strong nuclear force, respectively. In reality, the gauge group of the Standard Model takes the form

$$\mathrm{G}_{\mathrm{SM}} = \underbrace{\mathrm{U}(1)_Y \times \mathrm{SU}(2)}_{\text{electroweak}} \times \underbrace{\mathrm{SU}(3)}_{\text{strong}}. \tag{5.2.1}$$

The subscript on U(1) refers to the fact that we have not quite discovered electromagnetism yet; Y is *hypercharge*, which is a quantum number that for now acts as a replacement for standard electric charge Q.

In this proposed electroweak theory, we expect a massless gauge boson (the photon) in order to match with experiments. That is, we are looking for a way to spontaneously break the electroweak group into one of just electromagnetism:

$$\mathrm{U}(1)_Y \times \mathrm{SU}(2) \xrightarrow{SSB} \mathrm{U}(1)_{\mathrm{EM}}. \tag{5.2.2}$$

A quick dimension count says that four degrees of freedom (our group generators) going in split into three broken generators and one unbroken generator. The three broken generators will produce massless Goldstone modes that can be absorbed into weak force carriers and present us with the three massive gauge bosons. This is predicted precisely by Goldstone's theorem: the number of Goldstone modes should equal

$$\dim\left(\mathrm{U}(1)_Y \times \mathrm{SU}(2)\right) - \dim\left(\mathrm{U}(1)_{\mathrm{EM}}\right) = 4 - 1 = 3. \tag{5.2.3}$$

The final generator is unbroken, leaving one massless gauge boson to evade capture from the Higgs field: the photon. How might we want this to work? To answer that, we need the SU(2) algebra.

5.2.1 Applying the SU(2) algebra

We saw in Chapter 2 that a good choice for the $\mathfrak{su}(2)$ basis is $\{\mathrm{i}\sigma_a\}$ for Pauli's σ matrices and $a = 1, 2, 3$. They satisfy commutation relations (2.1.1)

$$[\sigma_a, \sigma_b] = 2\mathrm{i}\varepsilon_{abc}\sigma_c. \tag{5.2.4}$$

In what follows, it will be useful to switch to a renormalised basis in which we place a factor of 1/2 in front of the matrices in order to normalise the commutation relation:

$$\tau_a := \left\{ \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \implies [\tau_a, \tau_b] = \mathrm{i}\varepsilon_{abc}\tau_c. \tag{5.2.5}$$

This commutation relation is now in a form familiar to us when compared to what we wrote down for transformations in SU(N) (5.1.1); the structure constants have just been replaced by the Levi-Civita symbol, so we can play a similar game to understand gauge transformations of this group. For our analysis we will be considering doublets of SU(2), so keeping Pauli's basis is a sensible choice—they form 2D representations.

A local SU(2) transformation on a doublet $\boldsymbol{\phi}(x)$ takes the form

$$\boldsymbol{\phi} = \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \longmapsto \boldsymbol{\phi}' = \mathrm{e}^{\mathrm{i}g\alpha^a(x)\tau_a} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix}, \tag{5.2.6}$$

introducing the coupling strength g that tells us how strongly the SU(2) transformation acts on the fields. It is analogous to electric charge e that we saw previously.

Let us call this transformation $U(\boldsymbol{\alpha})$ for ease of notation, where $\boldsymbol{\alpha} = \alpha^a(x)\tau_a$. This formalism allows us to discuss how a gauge field $\boldsymbol{W}_\mu = W^a{}_\mu\tau_a$ may transform in this representation. From above we know this is under the adjoint (5.1.3), so the extra factors of U stick along for the ride:

$$\boldsymbol{W}_\mu \longmapsto U\left(\boldsymbol{W}_\mu + \frac{\mathrm{i}}{g}\partial_\mu\right)U^{-1} = U\left(\boldsymbol{W}_\mu + \partial_\mu\boldsymbol{\alpha}\right)U^{-1}. \tag{5.2.7}$$

For further analysis of these fields, especially in the context of electric charge (see Section 5.2.4), it will be beneficial to note the structure of infinitesimal transformations of the above gauge field. One can infinitesimally expand $U(\boldsymbol{\alpha})$ to first order in $\boldsymbol{\alpha}$ and observe:

$$\begin{aligned} U(\boldsymbol{\alpha}) \approx \mathbb{1} + \mathrm{i}g\alpha^a\tau_a \implies W^a{}_\mu &\longmapsto \left(\mathbb{1} + \mathrm{i}g\alpha^b\tau_b\right)\left(W^a{}_\mu + \partial_\mu\alpha^a\right)\left(\mathbb{1} - \mathrm{i}g\alpha^c\tau_c\right) \\ &\approx W^a{}_\mu + \partial_\mu\alpha^a - g\varepsilon^{abc}\alpha^b W^c{}_\mu. \end{aligned} \tag{5.2.8}$$

The associated field strength tensor is defined with the Levi-Civita structure constants so that

$$W^a{}_{\mu\nu} = \partial_\mu W^a{}_\nu - \partial_\nu W^a{}_\mu - g\varepsilon^{abc}W^b{}_\mu W^c{}_\nu, \tag{5.2.9}$$

which transforms under the adjoint also. This transformation leaves the trace of $(\boldsymbol{W}_{\mu\nu})^2$ invariant, meaning it belongs in any Lagrangian governed by an SU(2) field.

5.2.2 The Glashow-Weinberg-Salam model: a first look

Via Goldstone's theorem in Section 4.2.2 it turned out that any naïvely-built theory will give us massless particles, no matter the global group transformations. In order to make this unique to particles such as the photon we had to understand what occurs with gauge groups. Upon revising this theory with the Abelian Higgs model the photon was provided with a mass. As mentioned many a time, we need to re-revise the theory in order to fix this. This brings us to finally introducing the non-Abelian Higgs model in the context of electroweak symmetry breaking.

We discuss the dynamics of the Higgs particle H in the presence of a combined $\mathrm{U}(1)_Y \times \mathrm{SU}(2)$ gauge field. The $\mathrm{U}(1)_Y$ gauge field is relabelled as B_μ with its field strength tensor $B_{\mu\nu}$ and coupling strength $g'Y$, so as to stop the confusion with what we called the photon field earlier—remember, we are trying to rebuild an electromagnetic theory from something new. Moreover, we can include the hypercharge factor Y into this coupling strength which will mimic including electric charge e. In the specific case for the Higgs field we note that its hypercharge is $Y = 1/2$.

► Many texts disagree with what one should regard as the Higgs hypercharge. As highlighted by the `physics.stackexchange.com` user Shen [100], a quick Wikipedia search will tell you that what we call Y is in fact 1. But Peskin & Shroeder say $Y = 1/2$, and Srednicki says $Y = -1/2$ [23, 27]—and yet mathematical physics is completely unphased. All of this is to say that any convention may be used so long as one is consistent and does the analysis right!

Regardless of the relabelling, its gauge transformations remain the same as (4.3.1). A gauge field transforming under the adjoint of SU(2) (5.2.7) and its associated field strength tensor transformation (5.2.9) are as above, allowing us to define a covariant derivative:

$$\mathcal{D}_\mu H = \left(\partial_\mu - \mathrm{i}\frac{g'}{2}B_\mu - \mathrm{i}gW^a{}_\mu\tau_a\right)H. \tag{5.2.10}$$

The Higgs potential we can choose so that its ground state is nontrivial.

The non-Abelian Higgs model

The Lagrangian we form from the above is the **non-Abelian Higgs model**:

$$\mathscr{L} = -(\mathcal{D}_\mu H)^\dagger \mathcal{D}^\mu H - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}\mathrm{tr}(\boldsymbol{W}_{\mu\nu}\boldsymbol{W}^{\mu\nu}) - \lambda\left(H^\dagger H - \frac{\nu^2}{2}\right)^2, \tag{5.2.11}$$

studied by Glashow[a], Weinberg and Salam in various models that covered, in the main, interactions with hadrons and leptons [64, 78, 82].

[a] Sheldon Glashow, 1932 - , American, Nobel Prize 1979.

After understanding the necessary fermions more rigorously in Section 5.3 we can return to this model and understand how electroweak theory affects them. For now we can see how exactly the Higgs mechanism takes place in order to form gauge bosons. A more rigorous explanation of the following expansion is left to Appendix B since the procedure should be fairly familiar, but the main points one should take from it are left in. For this we begin with a Higgs field H that is now a doublet,

$$H := \begin{bmatrix} H^+ \\ H^0 \end{bmatrix}. \tag{5.2.12}$$

► H is redefined from the scalar field ϕ that appeared in our previous discussions. The indices are **not** completely arbitrary: under spontaneous symmetry breaking, the lower component of H is the one to receive a nonzero VEV and is electrically neutral. In contrast, the upper component receives a positive charge. This is more formally explained at the end of this section.

The doublet is a good choice for analysis because it matches the dimensions required for an electroweak transformation: H has four degrees of freedom from its two complex components, perfect for an electroweak transformation. One is always able to align coordinates so that the ground state of the

Higgs field $\langle H\rangle_0$ appears as a single component in the doublet:

$$\langle H\rangle_0 := \begin{bmatrix} 0 \\ \nu \end{bmatrix}, \tag{5.2.13}$$

Via the proposed symmetry breaking (5.2.2), it is evident that our manifold of ground states is a three-dimensional manifold from our broken generators. We can perturb this chosen ground state with an arbitrary 'angular' part parametrised by $\boldsymbol{\xi}(x) = \xi^a(x)\tau_a$ and a 'radial' part $\eta(x)$ as seen many a time:

$$H = \frac{1}{\sqrt{2}}\exp\{\mathrm{i}\xi^a(x)\tau_a\}\begin{bmatrix} 0 \\ \nu + \eta(x) \end{bmatrix} =: \frac{1}{\sqrt{2}}\mho\begin{bmatrix} 0 \\ \nu + \eta(x) \end{bmatrix}, \tag{5.2.14}$$

where for ease of notation we have replaced the broken symmetry transformation by $\mho$. One can interpret η as the Higgs boson, now with $\xi^a(x)$ as the three Goldstone modes ready to be eaten. We can see directly how this works via the Higgs mechanism. With this local perturbation in mind the covariant derivative in the above Lagrangian (5.2.11) is

$$\mathcal{D}_\mu H = \frac{1}{\sqrt{2}}\mho\Bigg(\begin{bmatrix} 0 \\ \partial_\mu\eta(x) \end{bmatrix} + \Bigg(\underbrace{\mho^{-1}\partial_\mu\mho - \mathrm{i}g\mho^{-1}W^a{}_\mu\tau_a\mho}_{\text{familiar!}} - \mathrm{i}\frac{g'}{2}B_\mu\Bigg)\begin{bmatrix} 0 \\ \nu + \eta(x) \end{bmatrix}\Bigg). \tag{5.2.15}$$

We have written it in this specific way because it allows us to realise the indicated combination looks similar to the gauge transformation (5.2.7). Note that $\mho$ and its inverse only appear in the context of a non-Abelian gauge transformation. This lets us revisit the idea of working in unitarity gauge, whereby we fix

$$W'^a{}_\mu := W^a{}_\mu - \frac{1}{g}\partial_\mu\xi^a(x). \tag{5.2.16}$$

In effect, we have set the phase parameters $\xi^a(x) = 0$ and hence $\mho = \mho^{-1} = \mathbb{1}$. From now on we will remove the prime on $\boldsymbol{W}'_\mu$ for the sake of notation, but note we are still in unitarity gauge. The ugly factors of $\mho$ and its inverse now vanish and the covariant derivative is reduced to

$$\mathcal{D}_\mu H = \frac{1}{\sqrt{2}}\Bigg(\begin{bmatrix} 0 \\ \partial_\mu\eta(x) \end{bmatrix} - \mathrm{i}\left(gW^a{}_\mu\tau_a + \frac{g'}{2}B_\mu\right)\begin{bmatrix} 0 \\ \nu + \eta(x) \end{bmatrix}\Bigg). \tag{5.2.17}$$

In order to make the picture more clear one can take a further step and actually expand out the matrix representations of the vector fields to then combine all similar terms into the $\underline{\mathbf{2}}$ of SU(2) as such:

$$gW^a{}_\mu\tau_a + \frac{g'}{2}B_\mu = \frac{1}{2}\begin{bmatrix} gW^3{}_\mu + g'B_\mu & g(W^1{}_\mu - \mathrm{i}W^2{}_\mu) \\ g(W^1{}_\mu + \mathrm{i}W^2{}_\mu) & -gW^3{}_\mu + g'B_\mu \end{bmatrix}. \tag{5.2.18}$$

We then insert this back into the covariant derivative terms, evaluate the matrix multiplication and simplify it down to reveal the kinetic Higgs term in the Lagrangian (5.2.11):

$$\begin{aligned}(\mathcal{D}_\mu H)^\dagger\mathcal{D}^\mu H &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta \\ &+ \frac{(\nu+\eta)^2}{8}\Big(g^2(W^1{}_\mu + \mathrm{i}W^2{}_\mu)(W^{1\mu} - \mathrm{i}W^{2\mu}) + (-gW^3{}_\mu + g'B_\mu)^2\Big).\end{aligned} \tag{5.2.19}$$

In a theory that once had massless particles, through studying a kinetic term involving the Higgs field, we have generated terms that correspond to massive particles that we can say lots about.

5.2.3 The photon mass: yes we can!

There now exist more massive fields in addition to the Higgs mass we gain from the potential, though in the current basis there is not a consistent way to extract them. In fact, by looking at the final term, it is evident that there is some non-trivial mixing between two of the fields that one might like to decode. By abstracting the four fields into a vector we can analyse its *mass matrix* that determines the strength of interactions between fields:

$$\text{mass terms} = \frac{\nu^2}{4}\begin{bmatrix} {W^1}_\mu \\ {W^2}_\mu \\ {W^3}_\mu \\ B_\mu \end{bmatrix}^\dagger \begin{bmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{bmatrix} \begin{bmatrix} {W^1}_\mu \\ {W^2}_\mu \\ {W^3}_\mu \\ B_\mu \end{bmatrix}. \tag{5.2.20}$$

Note the above matrix is block-diagonal, so we can treat the mixing in two separate cases. First, although the first block is completely diagonal already, it will be useful to redefine the W^1 and W^2 matrices into quantities that appear in the covariant derivative, so that

$${W^\pm}_\mu := \frac{1}{\sqrt{2}}({W^1}_\mu \mp \mathrm{i}{W^2}_\mu), \tag{5.2.21}$$

with the $\mp$ sign fixture to ensure the correct field has the corresponding electric charge ± 1 (5.2.36). Secondly, we can find the eigenvalues and eigenvectors of the lower-right matrix to realise

$$\frac{\nu^2}{4}\begin{bmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{bmatrix} \implies \text{eigenvalues } \left\{\frac{\nu^2}{4}(g^2+g'^2), 0\right\} \text{ \& eigenvectors } \begin{bmatrix} g \\ -g' \end{bmatrix}, \begin{bmatrix} g' \\ g \end{bmatrix}. \tag{5.2.22}$$

This tells us the nontrivial mixing of ${W^3}_\mu$ and B_μ can be broken into two separate fields: one massive and one massless. If we define a change of basis so that the lower-right matrix is of the form PDP^{-1}, where D is the diagonal matrix of eigenvalues, we can use the eigenvectors and define

$$P = \frac{1}{\sqrt{g^2+g'^2}}\begin{bmatrix} g & g' \\ -g' & g \end{bmatrix} \implies P^{-1} = \frac{1}{\sqrt{g^2+g'^2}}\begin{bmatrix} g & -g' \\ g' & g \end{bmatrix}, \tag{5.2.23}$$

which leads us directly to the exact mixing that provides the correct masses of the remaining gauge bosons:

$$\begin{bmatrix} Z_\mu \\ A_\mu \end{bmatrix} := P^{-1}\begin{bmatrix} {W^3}_\mu \\ B_\mu \end{bmatrix} = \frac{1}{\sqrt{g^2+g'^2}}\begin{bmatrix} g{W^3}_\mu - g'B_\mu \\ g'{W^3}_\mu + gB_\mu \end{bmatrix}. \tag{5.2.24}$$

This can be cleanly represented by what looks like a rotation matrix if we introduce Glashow's **weak mixing angle** [65] via

$$\cos\theta_W := \frac{g}{\sqrt{g^2+g'^2}},\ \sin\theta_W := \frac{g'}{\sqrt{g^2+g'^2}} \implies \begin{bmatrix} Z_\mu \\ A_\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{bmatrix}\begin{bmatrix} {W^3}_\mu \\ B_\mu \end{bmatrix}. \tag{5.2.25}$$

It was introduced for compact notations, but also to keep the choice of coupling strengths arbitrary between the weak and electromagnetic force and the gauge fields.

Vector bosons in the electroweak sector of the Standard Model

$$\begin{aligned} {W^+}_\mu &= \frac{1}{\sqrt{2}}({W^1}_\mu - \mathrm{i}{W^2}_\mu), \quad Z_\mu = \cos(\theta_W){W^3}_\mu - \sin(\theta_W)B_\mu, \\ {W^-}_\mu &= \frac{1}{\sqrt{2}}({W^1}_\mu + \mathrm{i}{W^2}_\mu), \quad A_\mu = \sin(\theta_W){W^3}_\mu + \cos(\theta_W)B_\mu. \end{aligned} \tag{5.2.26}$$

As for the full mass spectrum, there are five different particles we are talking about here: the **real** Higgs boson η, the three weak force carriers $W^+{}_\mu, W^-{}_\mu, Z_\mu$, and the photon A_μ. Each of their masses are given by quantities unified by gauge group coupling strengths, the Higgs parameter λ and, perhaps the most perculiar, the Higgs VEV ν—none of which we could have determined from the Lagrangian alone. They have all been found experimentally [54, 59, 71, 11].

	Predicted mass	Experimental value (GeV)
Higgs boson η	$\nu\sqrt{2\lambda}$	$m_\eta \approx 125.22$
W-boson $W^\pm{}_\mu$	$\frac{g\nu}{2}$	$m_{W^\pm{}_\mu} \approx 80.37$
Z-boson Z_μ	$\frac{\nu}{2}\sqrt{g^2+g'^2}$	$m_{Z_\mu} \approx 91.19$
Photon A_μ	0	$m_{A_\mu} < 5.61 \times 10^{-16}$

Table 5.2: Masses of bosons in the electroweak sector of the Standard Model.

If we wanted to, we could now extract precise values to the somewhat arbitrary coupling constants and Higgs VEV that were mathematically placed into the theory. The key point we are trying to make here is that the massless photon has been restored!

5.2.4 Electric charge

After understanding the process one should take a step back and ask if what we have produced is correctly named. We perhaps shouldn't take the apparition of a massless gauge boson as fundamental evidence, so does the unbroken group really correspond to electromagnetism?

Following a similar reasoning to Tong [49], one can consider a general electroweak transformation on the Higgs ground state. This is a simple action of matrix generators on $\langle H\rangle_0$, with couplings g and $g'Y$ as before:

$$\langle H\rangle_0 = \begin{bmatrix} 0 \\ \nu \end{bmatrix} \longmapsto \exp\{\mathrm{i}g\alpha^a\tau_a\}\exp\{\mathrm{i}g'\beta Y\}\begin{bmatrix} 0 \\ \nu \end{bmatrix}. \tag{5.2.27}$$

Note that $\boldsymbol{\alpha}$ and β are arbitrary parameters for our determination. We are acting specifically on the Higgs vacuum, so one notes that the hypercharge of the Higgs field is again $Y = 1/2$. The generators in the exponential can then be combined so that, when expanded, we have

$$\mathrm{i}g\alpha^a\tau_a + \mathrm{i}\frac{g'}{2}\beta = \frac{\mathrm{i}g}{2}\begin{bmatrix} \alpha^3 + g'\beta/g & \alpha^1 - \mathrm{i}\alpha^2 \\ \alpha^1 + \mathrm{i}\alpha^2 & -\alpha^3 + g'\beta/g \end{bmatrix}. \tag{5.2.28}$$

For an unbroken symmetry, this matrix generator should become trivial when acting on the Higgs vacuum $\langle H\rangle_0$ as with electromagnetism. This tells us all components of the generator other than the top left must vanish, since the three other components all interact with the Higgs VEV ν. This gives

$$\alpha^1 = \alpha^2 = 0, \quad \text{and} \quad \alpha^3 = \frac{g'\beta}{g}. \tag{5.2.29}$$

The transformation on a general state can now be reduced to the following:

$$\exp\{\mathrm{i}g\alpha^a\tau_a + \mathrm{i}g'\beta Y\} \longmapsto \exp\{\mathrm{i}g'\beta(\tau_3 + Y)\}. \tag{5.2.30}$$

This immediately gives some understanding as to why we expected nontrivial mixing between the individual elements of the electroweak group: we have a single generator in the matrix exponential that includes part of $\mathrm{U}(1)_Y$ and the third component of $\mathrm{SU}(2)$. Electric charge Q can now be defined as this unbroken generator, with τ_3 denoting the *weak isospin* of a doublet:

$$Q = \tau_3 + Y, \tag{5.2.31}$$

so that any matrix element $\mathcal{Q} \in \mathrm{U}(1)_{\mathrm{EM}}$ can be written as the exponential $\exp\{\mathrm{i}g'\beta Q\}$. From here, we can write infinitesimal transformations of the matrix charge operator $\mathcal{Q}$ acting on the Higgs as

$$\mathcal{Q} = \mathrm{e}^{\mathrm{i}g'\beta Q} \approx \mathbb{1} + \mathrm{i}g'\beta Q \implies \delta H = \mathrm{i}g'\beta Q H. \tag{5.2.32}$$

The charge of the Higgs field can then be calculated as this infinitesimal transformation acting on our doublet, replacing the generator as appropriate (5.2.31):

$$\delta H = \frac{\mathrm{i}g'\beta}{2}\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}H^+\\ H^0\end{bmatrix} = \mathrm{i}g'\beta\begin{bmatrix}H^+\\ 0\end{bmatrix}. \tag{5.2.33}$$

What is the punchline? Under the unbroken generator, which we deem electromagnetism,

$$QH^+ = +1 \cdot H^+ \ \text{ and } \ QH^0 = 0. \tag{5.2.34}$$

The Higgs field has a positively-charged component and a neutral component, justifying our choice of indices. We can also perform the same method to determine the charges of the gauge bosons, whereby we consider infinitesimal gauge transformations under the electroweak group and demand this is the same as infinitesimally transforming under $\mathcal{Q}$: multiplication by $\mathrm{i}g'\beta Q$.

The infinitesimal transformation of vector quantities are defined above (5.2.8), giving us the following:

$$\begin{aligned}\delta\left(W^{\pm}{}_{\mu}\right) &= \frac{1}{\sqrt{2}}\left(\delta(W^1{}_{\mu}) \mp \mathrm{i}\delta(W^2{}_{\mu})\right)\\ &= \frac{1}{\sqrt{2}}\Big(\partial_\mu\alpha^1 - g\varepsilon^{123}\alpha^2 W^3{}_{\mu} - g\varepsilon^{132}\alpha^3 W^2{}_{\mu}\\ &\qquad\qquad \mp \mathrm{i}\left(\partial_\mu\alpha^2 - g\varepsilon^{231}\alpha^3 W^1{}_{\mu} - g\varepsilon^{213}\alpha^1 W^3{}_{\mu}\right)\Big)\\ &= \pm\frac{1}{\sqrt{2}}\mathrm{i}g\alpha^3\left(W^1{}_{\mu} \mp \mathrm{i}W^2{}_{\mu}\right) = \pm\mathrm{i}g'\beta(W^{\pm}{}_{\mu}),\end{aligned} \tag{5.2.35}$$

remembering that we deduced the parameters $\alpha^1 = \alpha^2 = 0$ and $g\alpha^3 = g'\beta$. This leaves us to believe

$$QW^{\pm}{}_{\mu} = \pm W^{\pm}{}_{\mu}, \tag{5.2.36}$$

which again justifies their naming convention. As for the final two gauge bosons, note that under infinitesimal transformations the terms with epsilon factors will automatically vanish because they will contain a factor of either α^1 or α^2. This leaves us with only derivatives of α^3 and β which are, of course, zero, because they are constants:

$$\begin{aligned}\delta(Z_\mu) &= \cos(\theta_W)\partial_\mu\alpha^3 - \sin(\theta_W)\partial_\mu\beta = 0;\\ \delta(A_\mu) &= \sin(\theta_W)\partial_\mu\alpha^3 + \cos(\theta_W)\partial_\mu\beta = 0.\end{aligned} \tag{5.2.37}$$

Both Z_μ and A_μ are real-valued fields, since they are linear combinations of gauge fields, and so we could have come to this conclusion automatically (with similar reasoning for the Higgs boson η). But it is satisfying to see the same outcome algebraically.

5.3 Electroweak interactions with fermions

The way that quarks and leptons attain their masses in the Standard Model is, of course, via spontaneous symmetry breaking and subsequent coupling to the Higgs scalars. This is known as **Yukawa**[2] **coupling**. We do not have the time nor space to perform the calculations of this specific symmetry breaking with the right amount of justice, though the reader should now be familiar with the procedure regardless. For a more thourough discussion on the matter (!), the reader is pointed towards [45, p. 129-133], [49, p. 179-190, 202–204] and [40]. In the spacetime being we state the main results from the above references and close this chapter.

5.3.1 Quarks and leptons

We know that the Standard Model is a chiral theory [62, Abstract]; parity exchange is not a pure symmetry of the Standard Model, and, more specific to our study, left-chiral and right-chiral fermions transform differently under the electroweak symmetry group.

We mentioned previously that the pure SU(2) gauge interactions will only act on left-chiral fermions, meaning they all transform as doublets of SU(2). This contrasts to the right-chiral fermions, who must avoid interaction with the W and Z bosons, and so transform as *singlets* of SU(2). These results are consequences of the Wu[3] experiment [85, 86]. We now present the quarks and leptons in the Standard Model as in [45, p. 129], with left- and right-chirality indicated.

Three generations of each fermion

$$\begin{aligned} \mathcal{Q}_L^i &= \left\{ \begin{bmatrix} u_L \\ d_L \end{bmatrix}, \begin{bmatrix} c_L \\ s_L \end{bmatrix}, \begin{bmatrix} t_L \\ b_L \end{bmatrix} \right\}_{1/6} & \mathcal{U}_R^i &= \{u_R, c_R, t_R\}_{2/3} && \text{“ups”} \\ & & \mathcal{D}_R^i &= \{d_R, s_R, b_R\}_{-1/3} && \text{“downs”} \\ \mathcal{L}_L^i &= \left\{ \begin{bmatrix} \nu_{e,L} \\ e_L \end{bmatrix}, \begin{bmatrix} \nu_{\mu,L} \\ \mu_L \end{bmatrix}, \begin{bmatrix} \nu_{\tau,L} \\ \tau_L \end{bmatrix} \right\}_{-1/2} & \mathcal{N}_R^i &= \{\nu_{e,R}, \nu_{\mu,R}, \nu_{\tau,R}\}_0 && \text{“neutrinos”} \\ & & \mathcal{E}_R^i &= \{e_R, \mu_R, \tau_R\}_{-1} && \text{“electrons”} \end{aligned} \tag{5.3.1}$$

Each quark component $\begin{bmatrix} u_L^i & d_L^i \end{bmatrix} \in \mathcal{Q}_L^i$ and lepton component $\begin{bmatrix} \nu_L^i & e_L^i \end{bmatrix} \in \mathcal{L}_L^i$ will receive a weak isospin value of $\pm 1/2$ depending on its position in the doublet, since for each doublet we have the following action of τ_3:

$$\tau_3 \begin{bmatrix} u_L \\ d_L \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_L \\ d_L \end{bmatrix} = \frac{1}{2} \begin{bmatrix} +1 \cdot u_L \\ -1 \cdot d_L \end{bmatrix}, \tag{5.3.2}$$

using the up/down quark doublet as an example. The singlets do not interact with the weak bosons and therefore have a weak isospin of 0. This means that if each left-chiral and right-chiral fermion were to have the same electric charge, they must all have different hypercharges—this explains the subscripts on each doublet set above. Charges for each fermion[4] are then given as $Q = \tau_3 + Y$:

$$\begin{aligned} Q(u_L) &= \frac{1}{2} + \frac{1}{6} = +\frac{2}{3}, & Q(u_R) &= 0 + \frac{2}{3} = +\frac{2}{3}, \\ Q(d_L) &= -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}, & Q(d_R) &= 0 - \frac{1}{3} = -\frac{1}{3}, \\ Q(\nu_L) &= \frac{1}{2} - \frac{1}{2} = 0, & Q(\nu_R) &= 0 + 0 = 0, \\ Q(e_L) &= -\frac{1}{2} - \frac{1}{2} = -1, & Q(e_R) &= 0 - 1 = -1. \end{aligned} \tag{5.3.3}$$

[2] Hideki Yukawa, 1907 - 1981, Japanese, Nobel Prize 1949.
[3] Chien-Shiung Wu, 1912 - 1997, Chinese-American.
[4] The superscripts i are supressed for the sake of readability.

5.3.2 The Glashow-Weinberg-Salam model: a second look

The different hypercharges between each fermion component in (5.3.1) immediately forbids Lagrangian terms of the form

$$\overline{\Psi}\Psi = \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \tag{5.3.4}$$

for any left,right-chiral fermions ψ_L, ψ_R. We deem these *fermion mass terms* since they are proportional to the bilinears found in the Dirac field Lagrangian (3.2.5). In order to set ourselves up for symmetry breaking we take our GWS model (5.2.11) and reduce the terms to $\mathscr{L}_{Higgs}$ and $\mathscr{L}_{field\ strength}$ in an obvious manner. Then to it, we add kinetic terms describing the dynamics of the fermions using what we know about spinor Lagrangians (3.1.18), with the quarks as Ψ_q and leptons as Ψ_l:

$$\mathscr{L}_{kinetic} = \overline{\Psi}_q \mathrm{i}\gamma^\mu \mathcal{D}_\mu \Psi_q + \overline{\Psi}_l \mathrm{i}\gamma^\mu \mathcal{D}_\mu \Psi_l = \overline{\Psi}_q \mathrm{i}\not{\mathcal{D}}\Psi_q + \overline{\Psi}_l \mathrm{i}\not{\mathcal{D}}\Psi_l. \tag{5.3.5}$$

Or, using the projection operators (2.5.25) to separate the left- and right-chiral spinor fields,

$$\mathscr{L}_{kinetic} = \overline{\mathcal{Q}}^i_L \mathrm{i}\not{\mathcal{D}}\mathcal{Q}^i_L + \overline{\mathcal{L}}^i_L \mathrm{i}\not{\mathcal{D}}\mathcal{L}^i_L + \overline{\mathcal{U}}^i_R \mathrm{i}\not{\mathcal{D}}\mathcal{U}^i_R + \overline{\mathscr{D}}^i_R \mathrm{i}\not{\mathcal{D}}\mathscr{D}^i_R + \overline{\mathcal{N}}^i_R \mathrm{i}\not{\mathcal{D}}\mathcal{N}^i_R + \overline{\mathcal{E}}^i_R \mathrm{i}\not{\mathcal{D}}\mathcal{E}^i_R, \tag{5.3.6}$$

where we now use the Feynman slash to contract $\sigma^\mu \mathcal{D}_\mu$ or $\bar{\sigma}^\mu \mathcal{D}_\mu$ on each respective Weyl field. The final piece of the puzzle is to introduce the terms that involved Yukawa couplings. As remarked previously, we cannot dedicate enough space to properly study these, so the reader is directed to Section 6 of [49] for extra clarity. The general picture is that the bilinears (5.3.4) do not remain invariant under electroweak gauge transformations. What we require is some spectator field that will sit between the fields and counteract the transformation. The correct field to use would be the Higgs field H, transforming as an SU(2) doublet, so that (in an example of a coupling to the down quark):

$$\begin{aligned} \begin{bmatrix} u_L \\ d_L \end{bmatrix}^\dagger H d_R &\longmapsto \left(\mathrm{e}^{-\mathrm{i}g\alpha^a\tau_a}\mathrm{e}^{-\mathrm{i}g'\beta Y}\begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\right)\left(\mathrm{e}^{\mathrm{i}g\alpha^a\tau_a}H\right)\left(\mathrm{e}^{\mathrm{i}g'\beta Y}d_R\right) \\ &= \begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\begin{bmatrix}H^+ \\ H^0\end{bmatrix} d_R \xrightarrow{SSB} \begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\begin{bmatrix}0 \\ \nu\end{bmatrix} d_R = \bar{d}_L \nu d_R. \end{aligned} \tag{5.3.7}$$

► Note there also exists the hermitian-conjugated (hc) term $\bar{d}_R \widetilde{H} \boldsymbol{u}_L$ transforming in a similar manner. The Higgs H is replaced by its spinor conjugate, $\widetilde{H} := \begin{bmatrix}-\overline{H}^0 & \overline{H}^+\end{bmatrix}$.

Through spontaneous symmetry breaking we are therefore allowed to add gauge-invariant fermionic mass terms to our Lagrangians. Each of these terms will come with an attached **Yukawa coupling constant**, a set of matrices denoted by y^f_{ij} with a superscript dependent on which fermion it acts upon. This brings about the Yukawa sector of the electroweak Lagrangian:

$$\mathscr{L}_{Yukawa} = -y^u_{ij}\overline{\mathcal{Q}}^i_L \widetilde{H}\mathcal{U}^j_R - y^d_{ij}\overline{\mathcal{Q}}^i_L H\mathscr{D}^j_R - y^\nu_{ij}\overline{\mathcal{L}}^i_L \widetilde{H}\mathcal{N}^j_R - y^e_{ij}\overline{\mathcal{L}}^i_L H\mathcal{E}^j_R + \mathrm{hc}. \tag{5.3.8}$$

It is noted by Tong [49, p. 186] that because the coupling constants are matrices there exists lots of inter-generational mixing between the terms, which is certainly non-trivial.

Electroweak Lagrangian

With the separate Lagrangians provided previously, we present the full Lagrangian for the electroweak sector of the Standard Model:

$$\mathscr{L}_{EW} = \mathscr{L}_{Higgs} + \mathscr{L}_{field\ strength} + \mathscr{L}_{kinetic} + \mathscr{L}_{Yukawa}. \tag{5.3.9}$$

It may be expanded at the reader's peril.

6 Conclusion

6.1 Overview of the report

In order to understand what the Standard Model was describing we had to ensure that we knew how to deal with particles in a mathematical framework. This required us to study the geometric extension of Pauli's matrices and see that at the heart of physcial space lied an incredibly useful algebra. This could then be drawn directly to spacetime. We then spent a lot of time deriving what it meant for spacetime to be invariant under Poincaré transformations, which was all done to be able to define particles as **unitary irreducible representations of the Poincaré group**. Since we already discussed in previous courses how to transform integer-spin-valued objects we zoomed in on spinor-valued objects and discovered that all theories of fermions should be massless.

Leaving that strange feature behind momentarily, we realised that in order for special relativity to comply with quantum mechanics we required that our framework of particles must move to that of fields. We could then introduce the Lagrangian formalism of quantum field theory, which in turn unlocked internal symmetries and gauge invariances of the theory. This also predicted that any theory involving vector bosons should be massless too, which went against experimentation.

The final two chapters promised to solve the issue of the massless theories by understanding various ways in which to *break* the symmetries we learnt to love. With this, we were able to predict the number of massless bosons for any theory. The key to unlocking the massive theories, however, was to impose gauge invariance. This required a new particle, the Higgs boson, to interact with these fields and allowed us to predict the number of *massive* bosons, although this did accidentally predict a massive photon. The final fixture was to introduce electroweak interactions, which ticked all of the boxes: massless & massive boson & fermion theories were correctly predicted.

With infinite space and time the present author would have liked to discussed the electroweak interactions with fermions in more detail, as one recognises that a lot fo the results were stated with little intuitive reason behind them. Further, introducing the strong nuclear force as a gauge theory would have been welcomed. It is a force that acts only between quarks and can be mathematically described via the invariance of an SU(3) gauge group. Since SU(3) is eight-dimensional ($3^2 - 1 = 8$ from Section 5.1) there are eight force-carries that we denote the *gluons*. The study of how gluons interact with quarks is known as **quantum chromodynamics** (QCD), named for the fact that the SU(3) gauge invariance directly relates to *colour charge*. The reader is directed towards Chapter 6 of Osborn's notes and Chapters 6-9 of Georgi's book for further study [18, 43].

Another type of symmetry we did not get a chance to discuss was the famed CPT: the discrete symmetries of charge, parity and time, respectively. In their separate theories, there are some sectors of the Standard Model thay obey these symmetries. However, it is only when combined that the Standard Model shows its true beauty. This follows the notion that if all particles were switched for their antiparticle counterparts, all left-chiral particles were switched for their right-chiral counterparts, and time was reversed, the Standard Model would be completely invariant; the laws of the universe would remain the same. Two good discussions of this can be found in [45, p. 39-41] and [49, p. 36-46].

6.2 How does the Standard Model progress?

One might argue that since all of the particles from Figure 1.1 have been discovered, the Standard Model would be a complete theory. This is sadly not the case. For starters, it cannot be a theory of 'everything' insofar as fundamental forces go; the Standard Model does not predict anything about gravity, and we could have seen this ourselevs by the fact that our assumptions were based on *special* relativity. We said in Section 2.3 that gravity is the curvature of the spacetime field. If particles truly are the quanta of fields then hypothetically there should exist a particle that is the manifestation of ripples through spacetime: the elusive *graviton*. It might be sensible to suggest that the graviton fits in with the picture as in Figure 6.1, but an issue is that the Standard Model in incompatible with our current best theory of gravity, that being general relativity [80, p. 171101-1].

That being said, there does exist strong theoretical evidence that the graviton exists as a *tensor* boson through investigations of string theory [44, 32] and loop quantum gravity [34, 25], for example. They do, sadly, remain undetected as of today.

FIGURE 6.1: A hypothetical graviton that could be added to Figure 1.1. Inspired by [94].

There is also an entire study dedicated to physics beyond the Standard Model (BSM), in regards to attempting to solve problems that the Standard Model lacks the answers to. Some more open questions include [45, p. 18, 141–143]:

- Where does *dark matter* come from?
- Can we explain *dark energy*?
- Why do we observe *neutrino oscillations*, while the Standard Model does not predict them?

Some good sources to read that elaborate on these topics include [37, 74].

In summary, this report set out to detail the inner workings of the Standard Model from a group-theoretic point of view. This was motivated by the desire to understand what matter in the universe was made of and how certain kinds of matter should interact with other kinds of matter. In particular, we noted in the introduction that the Standard Model was birthed right when the theories of electroweak interactions were proven to be true, though one could argue its journey began with the first people who asked "what is *stuff* made of?". What we have done is taken ourselves on this journey under the, rather simple, assumption that the fundamental particles and interactions of the universe obey the laws of quantum mechanics and special relativity. This led us to:

- understanding what it means for particles to be constructed mathematically via representation theory;
- justifying how we can transfer this into a framework for fields;
- analysing spontaneous symmetry breaking; and
- uncovering electroweak interactions.

Perhaps we didn't contruct a theory of everything—but with each new day there is hope that theories going far beyond the Standard Model are one step closer to this goal. If and when we do test these theories and receive the results, whether they line up with our expectations or not, it is bound to change the course of mathematical physics forever.

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A
The Standard Model Lagrangian

Given locality, causality, Lorentz invariance and all known physical data since 1860, one can write down the Standard Model Lagrangian that describes all observed physical processes in the universe, ignoring gravity [96, 101, 29]:

$$
\begin{aligned}
\mathscr{L} =& -\frac{1}{2}\partial_\nu g^a{}_\mu \partial^\nu g^{a\mu} - g_s f_{abc}\partial_\mu g_\nu{}^a g^{b\mu} g^{c\nu} - \frac{1}{4} g_s^2 f_{abc} f^a{}_{de} g^b{}_\mu g^c{}_\nu g^{d\mu} g^{e\nu} - \partial_\nu W^+{}_\mu \partial^\nu W^{-\mu} - m_W^2 W^+{}_\mu W^{-\mu} \\
& -\frac{1}{2}\partial_\nu Z^0_\mu \partial^\nu Z^{0\mu} - \frac{m_W^2}{2c_w^2} Z^0_\mu Z^{0\mu} - \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2} m_H^2 H^2 + \partial_\mu \phi^+ \partial^\mu \phi^- - m_W^2 \phi^+ \phi^- \\
& + \frac{1}{2}\partial_\mu \phi^0 \partial^\mu \phi^0 - \frac{m_W}{2c_w^2}(\phi^0)^2 - \beta_H \left[\frac{2m_W^2}{g^2} + \frac{2m_W}{g} H + \frac{1}{2}\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right)\right] + \frac{2m_W^4}{g^2}\alpha_H \\
& - \mathrm{i} g c_w \Big[\partial_\nu Z^0_\mu \left(W^{+\mu} W^{-\nu} - W^{+\nu} W^{-\mu}\right) - Z^0_\nu \left(W^+_\mu \partial^\nu W^{-\mu} - W^-_\mu \partial^\nu W^{+\mu}\right) \\
& + Z^0_\mu \left(W^+_\nu \partial^\nu W^{-\mu} - W^-_\nu \partial^\nu W^{+\mu}\right)\Big] - \mathrm{i} g s_w \Big[\partial_\nu A_\mu \left(W^{+\mu} W^{-\nu} - W^{+\nu} W^{-\mu}\right) \\
& - A_\nu \left(W^+_\mu \partial^\nu W^{-\mu} - W^-_\mu \partial^\nu W^{+\mu}\right) + A_\mu \left(W^+_\nu \partial^\nu W^{-\mu} - W^-_\nu \partial^\nu W^{+\mu}\right)\Big] \\
& - \frac{1}{2} g^2 W^+_\mu W^{-\mu} W^+_\nu W^{-\nu} + \frac{1}{2} g^2 W^+_\mu W^-_\nu W^{+\mu} W^{-\nu} + g^2 c_w^2 \left(Z^0_\mu W^{+\mu} Z^0_\nu W^{-\nu} - Z^0_\mu Z^{0\mu} W^+_\nu W^{-\nu}\right) \\
& + g^2 s_w^2 \left(A_\mu W^{+\mu} A_\nu W^{-\nu} - A_\mu A^\mu W^+_\nu W^{-\nu}\right) + g^2 s_w c_w \Big[A_\mu Z^0_\nu \left(W^{+\mu} W^{-\nu} + W^{+\nu} W^{-\mu}\right) \\
& - 2 A_\mu Z^{0\mu} W^+_\nu W^{-\nu}\Big] - g \alpha_H m_W \left[H^3 + H(\phi^0)^2 + 2H\phi^+\phi^-\right] \\
& - \frac{1}{8} g^2 \alpha_h \Big[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2\Big] \\
& + g m_W W^+_\mu W^{-\mu} H + \frac{1}{2} g \frac{m_W}{c_w^2} Z^0_\mu Z^{0\mu} H + \frac{1}{2}\mathrm{i} g \Big[W^+_\mu \left(\phi^0 \partial^\mu \phi^- - \phi^- \partial^\mu \phi^0\right) \\
& - W^-_\mu \left(\phi^0 \partial^\mu \phi^+ - \phi^+ \partial^\mu \phi^0\right)\Big] - \frac{1}{2} g \Big[W^+_\mu \left(H \partial^\mu \phi^- - \phi^- \partial^\mu H\right) + W^-_\mu \left(H \partial^\mu \phi^+ - \phi^+ \partial^\mu H\right)\Big] \\
& + \frac{1}{2}\frac{g}{c_w} Z^0_\mu \left(H \partial^\mu \phi^0 - \phi^0 \partial^\mu H\right) + \mathrm{i} g \frac{s_w^2}{c_w} m_W Z^0_\mu \left(W^{+\mu}\phi^- - W^{-\mu}\phi^+\right) \\
& - \mathrm{i} g s_w m_W A_\mu \left(W^{+\mu}\phi^- - W^{-\mu}\phi^+\right) + \mathrm{i} g \frac{s_w^2 - c_w^2}{2c_w} Z^0_\mu \left(\phi^+ \partial^\mu \phi^- - \phi^- \partial^\mu \phi^+\right) \\
& - \mathrm{i} g s_w A_\mu \left(\phi^+ \partial^\mu \phi^- - \phi^- \partial^\mu \phi^+\right) + \frac{1}{4} g^2 W^+_\mu W^{-\mu} \left[H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right] + \ldots
\end{aligned}
$$

$$
\begin{aligned}
&\ldots + \frac{1}{8}\frac{g^2}{c_w^2} Z^0_\mu Z^{0\mu}\Big[H^2 + (\phi^0)^2 + 2(s_w^2 - c_w^2)\phi^+\phi^-\Big] + \frac{1}{2}g^2\frac{s_w^2}{c_w} Z^0_\mu \phi^0 \left[W^{+\mu}\phi^- + W^{-\mu}\phi^+\right] \\
&+ \frac{1}{2}\mathrm{i}g^2\frac{s_w^2}{c_w} Z^0_\mu H \left[W^{+\mu}\phi^- - W^{-\mu}\phi^+\right] - \frac{1}{2}g^2 s_w A_\mu \phi^0 \left[W^{+\mu}\phi^- + W^{-\mu}\phi^+\right] \\
&- \frac{1}{2}\mathrm{i}g^2 s_w A_\mu H \left[W^{+\mu}\phi^- - W^{-\mu}\phi^+\right] + g^2\frac{s_w}{c_w}(c_w^2 - s_w^2) A_\mu Z^{0\mu}\phi^+\phi^- + g^2 s_w^2 A_\mu A^\mu \phi^+\phi^- \\
&+ \bar{e}^\sigma(\mathrm{i}\gamma_\mu\partial^\mu - m_e^\sigma)e^\sigma + \bar{\nu}^\sigma \mathrm{i}\gamma_\mu\partial^\mu \nu\sigma + \bar{d}_j^\sigma(\mathrm{i}\gamma_\mu\partial^\mu - m_d^\sigma)d^{\sigma j} + \bar{u}_j^\sigma(\mathrm{i}\gamma_\mu\partial^\mu - m_u^\sigma)u^{\sigma j} \\
&+ g s_w A_\mu\Big[-(\bar{e}^\sigma\gamma^\mu e^\sigma) - \frac{1}{3}(d_j^\sigma\gamma^\mu d^{\sigma j}) + \frac{2}{3}(\bar{u}_j^\sigma\gamma^\mu u^{\sigma j})\Big] + \frac{g}{4c_w}Z^0_\mu\Big[\left(\bar{\nu}^\sigma\gamma^\mu(1-\gamma^5)\nu^\sigma\right) \\
&+ \left(\bar{e}^\sigma\gamma^\mu\left(4s_w^2 - (1-\gamma^5)\right)e^\sigma\right) + \left(\bar{d}_j^\sigma\gamma^\mu\left(\frac{4}{3}s_w^2 - (1-\gamma^5)\right)d^{\sigma j}\right) + \left(\bar{u}_j^\sigma\gamma^\mu\left(-\frac{8}{3}s_w^2 + (1-\gamma^5)\right)u^{\sigma j}\right)\Big] \\
&+ \frac{g}{2\sqrt{2}}W^+_\mu\Big[\left(\bar{\nu}^\sigma\gamma^\mu(1-\gamma^5)P^{\sigma\tau}e^\tau\right) + \left(\bar{u}_j^\sigma\gamma^\mu(1-\gamma^5)C^{\sigma\tau}d^{\tau j}\right)\Big] \\
&+ \frac{g}{2\sqrt{2}}W^-_\mu\Big[\left(\bar{e}^\sigma\gamma^\mu(1-\gamma^5)P^{\dagger\sigma\tau}\nu^\tau\right) + \left(\bar{d}_j^\sigma\gamma^\mu(1-\gamma^5)C^{\dagger\sigma\tau}u^{\tau j}\right)\Big] \\
&+ \mathrm{i}\frac{g}{s\sqrt{2}}\frac{m_e^\sigma}{m_W}\Big[-\phi^+\left(\bar{\nu}^\sigma(1+\gamma^5)e^\sigma\right) + \phi^-\left(\bar{e}^\sigma(1-\gamma^5)\nu^\sigma\right)\Big] - \frac{g}{2}\frac{m_e^\sigma}{m_W}\left[H\bar{e}^\sigma e^\sigma - \mathrm{i}\phi^0\bar{e}^\sigma\gamma^5 e^\sigma\right] \\
&+ \mathrm{i}\frac{g}{2\sqrt{2}m_W}\phi^+\Big[-m_d^\tau\left(\bar{u}_j^\sigma C^{\sigma\tau}(1+\gamma^5)d^{\tau j}\right) + m_u^\tau\left(\bar{u}_j^\sigma C^{\sigma\tau}(1-\gamma^5)d^{\tau j}\right)\Big] \\
&+ \mathrm{i}\frac{g}{2\sqrt{2}m_W}\phi^-\Big[m_d^\tau\left(\bar{d}_j^\sigma C^{\dagger\sigma\tau}(1-\gamma^5)u^{\tau j}\right) - m_u^\tau\left(\bar{d}_j^\sigma C^{\dagger\sigma\tau}(1+\gamma^5)u^{\tau j}\right)\Big] \\
&- \frac{g}{2}\frac{m_u^\sigma}{m_W}H\bar{u}_j^\sigma u^{\sigma j} - \frac{g}{2}\frac{m_d^\sigma}{m_W}H\bar{d}_j^\sigma d^{\sigma j} - \mathrm{i}\frac{g}{2}\frac{m_u^\sigma}{m_W}\phi^0\bar{u}_j^\sigma\gamma^5 u^{\sigma j} + \mathrm{i}\frac{g}{2}\frac{m_d^\sigma}{m_W}\phi^0\bar{d}_j^\sigma\gamma^5 d^{\sigma j} \\
&- \frac{1}{2}\mathrm{i}g_s\bar{d}_i^\sigma\gamma_\mu\lambda_a^{ij}d^{\sigma j}g^{a\mu} - \frac{1}{2}\mathrm{i}g_s\bar{u}_i^\sigma\gamma_\mu\lambda_a^{ij}u^{\sigma j}g^{a\mu} \\
&- \overline{X}^+\left(\partial_\mu\partial^\mu + m_W^2\right)X^+ - \overline{X}^-\left(\partial_\mu\partial^\mu + m_W^2\right)X^- - \overline{X}^0\left(\partial_\mu\partial^\mu + \frac{m_W^2}{c_w^2}\right)X^- - \overline{Y}\partial_\mu\partial^\mu Y \\
&- \mathrm{i}gc_w W^+_\mu\left(\partial^\mu\overline{X}^0 X^- - \partial^\mu\overline{X}^+ X^0\right) - \mathrm{i}gs_w W^+_\mu\left(\partial^\mu\overline{Y}X^- - \partial^\mu\overline{X}^+ Y\right) \\
&- \mathrm{i}gc_w W^-_\mu\left(\partial^\mu\overline{X}^- X^0 - \partial^\mu\overline{X}^0 X^+\right) - \mathrm{i}gs_w W^-_\mu\left(\partial^\mu\overline{X^-}Y - \partial^\mu\overline{Y}X^+\right) \\
&- \mathrm{i}gc_w Z^0_\mu\left(\partial^\mu\overline{X}^+ X^+ - \partial^\mu\overline{X}^- X^-\right) - \mathrm{i}gs_w A_\mu\left(\partial^\mu\overline{X}^+ X^+ - \partial^\mu\overline{X}^- X^-\right) \\
&- \frac{1}{2}gm_W\left[\overline{X}^+ X^+ H + \overline{X}^- X^- H + \frac{1}{c_w^2}\overline{X}^0 X^0 H\right] \\
&+ \frac{s_w^2 - c_w^2}{2c_w}\mathrm{i}gm_W\left[\overline{X}^+ X^0\phi^+ - \overline{X}^- X^0\phi^-\right] + \frac{1}{2c_w}\mathrm{i}gm_W\left[\overline{X}^0 X^-\phi^+ - \overline{X}^0 X^+\phi^-\right] \\
&+ \mathrm{i}gm_W s_w\left[\overline{X}^- Y\phi^- - \overline{X}^+ Y\phi^+\right] + \mathrm{i}\frac{1}{2}gm_W\left[\overline{X}^+ X^+\phi^0 - \overline{X}^- X^-\phi^0\right] \\
&- \overline{G}_a\partial_\mu\partial^\mu G^a - g_s f_{abc}\partial_\mu\overline{G}^a G^b g^{c\mu}.
\end{aligned}
$$

B

Simplifying the Kinetic Higgs Term in Glashow-Weinberg-Salam Theory

We aim to expand the kinetic Higgs term in the following Lagrangian after perturbation around a chosen ground state:

$$\mathscr{L} = -\underbrace{(D_\mu H)^\dagger D^\mu H}_{\text{this one}} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}\mathrm{tr}(\boldsymbol{W}_{\mu\nu}\boldsymbol{W}^{\mu\nu}) - \lambda\left(H^\dagger H - \frac{\nu^2}{2}\right)^2. \tag{B.0.1}$$

The covariant derivative and its Hermitian conjugate are

$$D_\mu H = \left(\partial_\mu - \mathrm{i}\frac{g'}{2}B_\mu - \mathrm{i}gW^a{}_\mu\tau_a\right)H,\ (D_\mu H)^\dagger = \left(\partial_\mu + \mathrm{i}\frac{g'}{2}B_\mu + \mathrm{i}gW^a{}_\mu\tau_a\right)H^\dagger. \tag{B.0.2}$$

We can align coordinates so that the ground state of the Higgs field is

$$\langle H\rangle_0 = \begin{bmatrix}0\\ \nu\end{bmatrix}, \tag{B.0.3}$$

and a perturbation around this ground state is given by

$$H = \frac{1}{\sqrt{2}}\exp\{\mathrm{i}\xi^a(x)\tau_a\}\begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix} =: \frac{1}{\sqrt{2}}\mho\begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix}, \tag{B.0.4}$$

where for ease of notation I have replaced the broken symmetry transformation by $\mho$. The covariant derivative now becomes

$$\begin{aligned}
D_\mu H &= \frac{1}{\sqrt{2}}\left(\partial_\mu\mho\begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix} + \mho\begin{bmatrix}0\\ \partial_\mu\eta(x)\end{bmatrix}\right) - \mathrm{i}\left(\frac{g'}{2}B_\mu + gW^a{}_\mu\tau_a\right)\mho\begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix}\\
&= \frac{1}{\sqrt{2}}\mho\Bigg(\begin{bmatrix}0\\ \partial_\mu\eta(x)\end{bmatrix} + \left(\mho^{-1}\partial_\mu\mho - \mathrm{i}\frac{g'}{2}\mho^{-1}B_\mu\mho - \mathrm{i}g\mho^{-1}W^a{}_\mu\tau_a\mho\right)\begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix}\Bigg)\\
&= \frac{1}{\sqrt{2}}\mho\Bigg(\begin{bmatrix}0\\ \partial_\mu\eta(x)\end{bmatrix} + \left(\mho^{-1}\partial_\mu\mho - \mathrm{i}g\mho^{-1}W^a{}_\mu\tau_a\mho - \mathrm{i}\frac{g'}{2}B_\mu\right)\begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix}\Bigg).
\end{aligned} \tag{B.0.5}$$

Similarly for its Hermitian conjugate,

$$(D_\mu H)^\dagger = \frac{1}{\sqrt{2}}\Bigg(\begin{bmatrix}0\\ \partial_\mu\eta(x)\end{bmatrix}^\dagger + \begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix}^\dagger\left((\partial_\mu\mho^{-1})\mho + \mathrm{i}g\mho^{-1}W^a{}_\mu\tau_a\mho + \mathrm{i}\frac{g'}{2}B_\mu\right)\Bigg)\mho^{-1}. \tag{B.0.6}$$

Note that $\mho$ and its inverse only appear in the context of a gauge transformation. We can now revisit the idea of working in unitarity gauge. This effectively sets the phase parameters $\xi^a(x) = 0$ and hence $\mho = \mho^{-1} = \mathbb{1}$. In unitary gauge, the covariant derivatives become

$$\begin{aligned}
D_\mu H &= \frac{1}{\sqrt{2}}\Bigg(\begin{bmatrix}0\\ \partial_\mu\eta(x)\end{bmatrix} - \mathrm{i}\left(gW^a{}_\mu\tau_a + \frac{g'}{2}B_\mu\right)\begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix}\Bigg);\\
(D_\mu H)^\dagger &= \frac{1}{\sqrt{2}}\Bigg(\begin{bmatrix}0\\ \partial_\mu\eta(x)\end{bmatrix}^\dagger + \mathrm{i}\begin{bmatrix}0\\ \nu+\eta(x)\end{bmatrix}^\dagger\left(gW^a{}_\mu\tau_a + \frac{g'}{2}B_\mu\right)\Bigg).
\end{aligned} \tag{B.0.7}$$

We now expand out the matrix representations of the vector fields and combine all similar terms into the $\underline{\mathbf{2}}$ of $SU(2)$ as such:

$$\begin{aligned} W^a{}_\mu \tau_a &= \frac{1}{2}\begin{bmatrix} W^3{}_\mu & W^1{}_\mu - \mathrm{i}W^2{}_\mu \\ W^1{}_\mu + \mathrm{i}W^2{}_\mu & -W^3{}_\mu \end{bmatrix}; \\ \implies gW^a{}_\mu \tau_a + \frac{g'}{2}B_\mu &= \frac{1}{2}\begin{bmatrix} gW^3{}_\mu + g'B_\mu & g(W^1{}_\mu - \mathrm{i}W^2{}_\mu) \\ g(W^1{}_\mu + \mathrm{i}W^2{}_\mu) & -gW^3{}_\mu + g'B_\mu \end{bmatrix}. \end{aligned} \tag{B.0.8}$$

Multiplying this out into the covariant derivatives,

$$\begin{aligned} D_\mu H &= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 0 \\ \partial_\mu \eta(x) \end{bmatrix} - \frac{\mathrm{i}}{2}\begin{bmatrix} gW^3{}_\mu + g'B_\mu & g(W^1{}_\mu - \mathrm{i}W^2{}_\mu) \\ g(W^1{}_\mu + \mathrm{i}W^2{}_\mu) & -gW^3{}_\mu + g'B_\mu \end{bmatrix}\begin{bmatrix} 0 \\ \nu + \eta(x) \end{bmatrix} \right); \\ (D_\mu H)^\dagger &= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 0 \\ \partial_\mu \eta(x) \end{bmatrix}^\dagger + \frac{\mathrm{i}}{2}\begin{bmatrix} 0 \\ \nu + \eta(x) \end{bmatrix}^\dagger \begin{bmatrix} gW^3{}_\mu + g'B_\mu & g(W^1{}_\mu - \mathrm{i}W^2{}_\mu) \\ g(W^1{}_\mu + \mathrm{i}W^2{}_\mu) & -gW^3{}_\mu + g'B_\mu \end{bmatrix} \right). \end{aligned} \tag{B.0.9}$$

This implies

$$\begin{aligned} D_\mu H &= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 0 \\ \partial_\mu \eta(x) \end{bmatrix} - \frac{\mathrm{i}}{2}(\nu + \eta)\begin{bmatrix} g(W^1{}_\mu - \mathrm{i}W^2{}_\mu) \\ -gW^3{}_\mu + g'B_\mu \end{bmatrix} \right); \\ (D_\mu H)^\dagger &= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 0 \\ \partial_\mu \eta(x) \end{bmatrix}^\dagger + \frac{\mathrm{i}}{2}(\nu + \eta)\begin{bmatrix} g(W^1{}_\mu - \mathrm{i}W^2{}_\mu) \\ -gW^3{}_\mu + g'B_\mu \end{bmatrix}^\dagger \right). \end{aligned} \tag{B.0.10}$$

The kinetic Higgs term we are interested in now reads off as

$$\begin{aligned} (D_\mu H)^\dagger D^\mu H &= \frac{1}{2}\partial_\mu \eta \partial^\mu \eta + \frac{1}{8}(\nu + \eta)^2 \begin{bmatrix} g(W^1{}_\mu - \mathrm{i}W^2{}_\mu) \\ -gW^3{}_\mu + g'B_\mu \end{bmatrix}^\dagger \begin{bmatrix} g(W^{1\mu} - \mathrm{i}W^{2\mu}) \\ -gW^{3\mu} + g'B^\mu \end{bmatrix} \\ &= \frac{1}{2}\partial_\mu \eta \partial^\mu \eta + \frac{1}{8}(\nu + \eta)^2 \left(g^2 (W^1{}_\mu)^2 + g^2 (W^2{}_\mu)^2 + (-gW^3{}_\mu + g'B_\mu)^2 \right), \end{aligned} \tag{B.0.11}$$

with the cross terms cancelling out.