24 Hours of Le Maths 2026
Question Paper

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28 February – 1 March 2026

Welcome to 24 Hours Of Le Maths 2026!

The questions for this competition are presented in the following document. There are 645 for you to attempt, giving a total of 7140 marks for you to attain, plus 1402 marks arising from bonus potentials. Essentially, plenty for you to be getting on with!

When submitting your answers, please write the question code (of the form AB123) followed by your given answer. All sheets of paper handed in must have your team name on them. Please do not surround your answers with workings, as this makes the lives of Markers very difficult! *Marks are rewarded for correct answers only.*

If you have any questions, please ask any of the Volunteers at any time. *Please note this does not include the questions on the paper.*

Good luck!

Rules and Regulations

Any good event has rules. Any breakage of the rules will result in a penalty of marks by any of the associated Volunteers, who will be a collection of Markers, Question Writers and general people there to help.

The biggest, most important rule is this.

The use of calculators (which is any tool used to skip a part of the computational or thinking process), both online and physical, is banned. Any team using a calculator will be immediately disqualified.

Basically, don't cheat. It ruins the game. This competition is not about getting the marks; it's about problem solving, having fun and seeing if you can make it to the end! It's not that serious!

Scribes

- Teams may only have one scribe at a time; this includes writing down all of the working out and answers. Teammates are allowed to communicate with their scribe, but only the current scribe can write out workings or answers.
- Scribes must be in place for at least 15 minutes at a time before being swapped. They are also only allowed to be on shift for up to 4 hours (unless solo) at a time with a 30-minute cool-down period between shifts.
- In teams of 4, every team member must scribe for at least 3 hours total and no more than 8 hours. In teams of 3, every team member must scribe for at least 4 hours total and no more than 10 hours. In teams of 2, every team member must scribe for at least 6 hours total and no more than 18 hours.

Submitting answers

- All answers should be submitted with their corresponding marking code in the form AB123. Hand the answer sheet to a Marker.
- If an answer is correct, it will be added to your score. If the answer is incorrect, it will not **and you will not be allowed to reattempt the question.**
- If you feel an answer is unjustly marked as incorrect, you may appeal with a Volunteer who will dispute and raise the claim with the corresponding Question Writer.
- Any of the team members may hand in the answer sheet—but the answers themselves must be written by a scribe.
- Teams may hand in answers as often as they like but there must be at least one answer sheet per hour.
- When there is a scribe change, all answers must be handed in by the outgoing scribe at that moment, and they are to inform a Volunteer or Marker of the shift change.

Code of conduct

- Team members are not expected to be in exam conditions. In fact, we encourage chatter! However, you are not allowed to disturb other teams excessively and/or unnecessarily.
- The use of mobile phones, laptops and other devices is permitted for recreational purposes only.
- You must treat all Volunteers with respect.
- Food and drink may not be consumed in the Scott Logic lecture hall, but may be enjoyed elsewhere in the building.
- **Sleeping on-site is not allowed.** Either power through or return home to your accommodation or other private space for this.
- Team members are allowed to leave the event room at any time, assuming they are not the scribe, and for any reason. There is no need to sign in or out. A scribe may leave the room but not for accessing prohibited information.

Bonus features

Bonus marks

Each section has its own defined large number of points, and it is highly unlikely that you will be able to attain them all. It's okay—it's part of the process. What we *can* offer you are bonus marks. **If you correctly answer 75% of a section, you will be rewarded with a further 25% of the possible marks for that section.**

For example, say the Probability & Statistics section is worth 350 marks. If you have attained 263 of these marks, then you get an automatic 87 bonus marks. Groovy.

Lecturer questions

Once per hour, on the hour, starting a few hours into the event, a Lecturer Question will be released.

These questions have been submitted by one of the lecturers in the department, which means they can range from anything they taught you in first year up to what they are actively researching. Despite this, **all questions are able to be answered using first-year techniques.** You may just need to think a little harder!

Each question is worth 100 points, but those points will only go to the first team that submits the answer. If you have attained the points for a Lecturer Question, you may not enter an answer for the next Lecturer Question for the following hour.

Attributions

At the end of each section, there will be an attribution to the Question Writer which will let you know a little bit about them and give you ways to contact them.

Say thanks! Tell them how much you loved/hated/were confused by their questions! But, of course, please be respectful. They have all spent countless hours fussing over this question paper to make it look nice.

Notation and Definitions

Please note the following conventions of notation often used throughout.

- The Kronecker delta is defined as

$$\delta_{ij} := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

- You should note the following conventions when it comes to derivatives.
  - We may write $f'(\cdot)$ for the derivative of a function with respect to its argument. f'' and f''' denote the second and third derivatives, beyond which we write $f^{(n)}$ for the $n^{\underline{\text{th}}}$ derivative.
  - If $f(t)$ is a function of *time*, then we may write

$$\dot{f}(t) \equiv \frac{\mathrm{d}f}{\mathrm{d}t}, \text{ and } \ddot{f}(t) \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}$$

    for the first and second derivatives with respect to time, respectively.
  - When working with partial derivatives, we may, for concision, write

$$\partial_x f \equiv \frac{\partial f}{\partial x} \quad \text{or} \quad \partial_i f \equiv \frac{\partial f}{\partial x_i}.$$

- You should note the following conventions when it comes to multi-varietal derivatives.
  - The differential operator ∇ (named 'nabla' or 'del') is defined in the Cartesian coordinate system of $\mathbb{R}^n$ with standard basis by

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right).$$

  - Del acts on **scalar-valued** functions $f(x_1, \ldots, x_n)$ via the *gradient*:

$$\nabla f = (\partial_1 f, \partial_2 f, \ldots, \partial_n f).$$

  - Del acts on **vector-valued** functions $\boldsymbol{u}(x_1, \ldots, x_n)$ via the *divergence*:

$$\nabla \cdot \boldsymbol{u} := \partial_1 u_1 + \partial_2 u_2 + \cdots + \partial_n u_n,$$

    and the *curl* in three dimensions only:

$$\nabla \times \boldsymbol{u} := \det \begin{bmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{bmatrix}.$$

- We may use the following notation to denote the binomial coefficients:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!},$$

or en notación española,

$$\binom{n}{k} := \frac{¡n!}{¡k!\,¡(n-k)!}.$$

- Unless otherwise stated, log will denote the natural logarithm, and *not* a logarithm of base 10. This should cause no confusion as we will not be considering complex logarithms nor using logarithms of base 10. We have standards.
- Unless otherwise stated, all angles should be given in radians.
- We define the floor and ceiling functions by

$$\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z} : x \mapsto \lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\},$$
$$\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z} : x \mapsto \lceil x \rceil := \min\{n \in \mathbb{Z} \mid n \geq x\},$$

respectively. We define the fractional part by

$$\{\cdot\} : \mathbb{R} \to [0,1) : x \mapsto \{x\} := x - \lfloor x \rfloor.$$

- We consider 1 not to be a prime number. (But in a sense, it's a bit of a definition).
- You may wish to recall the definition of the gamma function:

$$\Gamma : \{z \in \mathbb{C} \mid \Re(z) > 0\} \to \mathbb{C} : z \mapsto \Gamma(z) := \int_0^\infty t^{z-1} \mathrm{e}^{-t} \,\mathrm{d}t,$$

which satisfies $\Gamma(n) = (n-1)!$ and thus $z\Gamma(z) = \Gamma(z+1)$.
- The Riemann zeta function is defined as:

$$\zeta : \{s \in \mathbb{C} \mid \Re(s) > 1\} \to \mathbb{C} : s \mapsto \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

⟦*Note*: these aren't quite the definitions of the gamma and Riemann zeta functions. Indeed, the gamma and zeta functions are in fact the analytic continuations of the above functions to $\mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ and $\mathbb{C} \setminus \{1\}$. If you've taken Complex Analysis II, this will make sense. If you didn't take Complex Analysis II, you can ignore all this and treat them just as they are defined here.⟧

24h LE MATHS 2026

The question paper, its solutions and the full list of lecturer questions will be released for your enjoyment after the event. Should you wish to use a question for publishing or sharing, ensure you give correct attribution to the writer(s).

Arithmetic

Welcome to the **Arithmetic** section!

As this is (most likely) the first section you stumble upon, it would be wise to make a good first impression. I hope that the problems and puzzles that follow spark joy. After all, I've had a lot of fun designing them!

Answer as many questions as you wish via any viable method and please simplify them as much as possible.

Keep it groovy! ☙

~ *Sam Kay*

AR001. $1 + 1$

[1]

AR002. $1 + 1$, before the publication of *Principia Mathematica* by Whitehead and Russell in 1910.

[1]

AR003. $865 - 239$

[1]

AR004. Find the sum of the digits in 80085.

[1]

AR005. $34 + 35$

[1]

AR006. If I had 200 Yorkshire Tea tea bags at the start of term and have since used 157 of them, how many remain?

[1]

AR007. 2×187

[1]

AR008. 46×64

[2]

AR009. $(28 + 02) - (01 + 03) - (20 + 26)$

[2]

AR010. How many (integer number of) coffees can I make from my 250 g bag of Colombian single origin beans if I weighed each cup using 15 g of beans?

[2]

AR011. What if each cup uses 18 g of beans?

[2]

AR012. BIDMAS RULES!

Evaluate $7(3+1) - 4 \times 5 + 2$.

[2]

AR013. Wait... what does BIDMAS stand for again?

[2]

AR014. $\sqrt{289}$

[2]

AR015. $\sqrt{0.81} + \sqrt{0.49}$

[3]

AR016. $\sqrt{\sqrt{0.25} + \sqrt{1.44} - 0.01}$

[3]

AR017. 271×828

[3]

AR018. $\frac{5}{9} - \frac{3}{25}$

[3]

AR019. Write $\frac{1013}{50}$ as a decimal.

[3]

AR020. 9^5

[3]

AR021. $|4 - |3 - 7|| + |11 - |5 - 18| + 2|$

[3]

AR022. 17×5882353

[4]

AR023. $30081 \div 37$

[4]

AR024. $14892384 - 2216489$

[4]

AR025. $345.24 + 231.56$

[4]

AR026. $2^2 + 0^{0!} + 2^2 + 6^6$

[4]

AR027. Find a pair of numbers a, b in the following set such that $7 \mid a - b$:

$$S = \{75, 65, 13, 7, 71, 82, 72, 19\}.$$

[4]

AR028. Find a pair of numbers c, d in the following set such that $9 \mid c - d$:

$$T = \{55, 71, 2, 97, 89, 82, 18, 3, 38, 93\}.$$

[4]

AR029. Does there exist two distinct pairs of numbers $\{e, f\}, \{g, h\}$ in the following set such that $19 \mid e - f$ and $21 \mid g - h$?

$$U = \{49, 6, 43, 27, 53, 55, 78, 80, 83, 91, 35, 41, 68, 70, 47, 86, 10, 8\}.$$

[5]

AR030. $\sqrt{3969}$

[5]

AR031. $3^3 + 4^4 + 3^3 + 5^5$

[5]

AR032. $3 + \frac{14}{15} + \frac{926}{535}$ to six decimal places.

[5]

AR033. 9804×6699

[5]

AR034. $3! \times 4!$

[5]

AR035. $\sqrt{441 + 1302 + 961}$

[5]

AR036. How much egg you fry

If Stephen has six eggs and 100 pieces of bacon, what is the maximum number of bacon and egg sandwiches he can make if each sandwich contains half an egg and eight pieces of bacon?

[6]

AR037. 12345×54321

[6]

AR038. $73654 + 282595 + 18311460 + 59193865$

[6]

AR039. Evaluate $1 + 2 + \ldots + 10$.

[6]

AR040. Evaluate $1 + 2 + \ldots + 100$.

[6]

AR041. Evaluate $1 + 2 + \ldots + 1000$.

[6]

AR042. Evaluate $(-26) + (-25) + \ldots + 26$.

[6]

AR043. Evaluate $(-26) - (-25) + (-24) - \ldots + 26$.

[7]

AR044. $7!$

[7]

AR045. $10!$

[7]

AR046. What is the second smallest integer that is divisible by each number 1 through 10?

[7]

AR047. How many seconds are there in 6 weeks?

[7]

AR048. Evaluate $-10 \times -9 \times \cdots \times 9 \times 10$.

[7]

AR049. $98764125364 - 981765425602$

[8]

AR050. $12! \times 4!$

[8]

AR051. Simplify $\sqrt{11875}$

[8]

AR052. $0.000000000000000000000009 + 0.00000000000000000000000001$

[8]

AR053. Convert 1001001001010 (base 2) into base 3.

[8]

AR054. $2 + \frac{71}{82} + \frac{818}{284} + \frac{5904}{5235}$ to 10 decimal places.

[8]

AR055. $1 + 11 + 111 + 1111 + 11111 + 111111 + 1111111 + 11111111 + 111111111$

[9]

AR056. $1 + 22 + 333 + 4444 + 55555 + 666666 + 7777777 + 88888888 + 999999999$

[9]

AR057. $99999999 - 8888888 + 777777 - 66666 + 5555 - 444 + 33 - 2$

[9]

AR058. Convert 100000001 (base 2) + 100000001 (base 3) into base 4.

[9]

AR059. $1 - \frac{1}{11} + \frac{1}{111} - \frac{1}{1111} + \frac{1}{11111}$ to 11 decimal places.

[9]

AR060. $\sqrt[3]{2002 + 17 + 1717 + 1177}$

[9]

AR061. The blue colour you see before you has the HEX code #2E75FF. What is this in base 10?

[10]

AR062. I know this isn't how colours work but this seemed fun okay?

The colour on the Durham University logo (sometimes called 'Palatinate') can be represented by 4653199 in base 10. What is its HEX code?

[10]

AR063. $757421395732 \times 391571003215$

[10]

AR064. $12500580262 \div 137$

[10]

AR065. $1^7 + 4^7 + 4^7 + 5^7 + 9^7 + 9^7 + 2^7 + 9^7$

[10]

AR066. Given that this converges, evaluate

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}}$$

[10]

> **Six Seven**
>
> Each team that provides the total number of questions in this entire document that contain both a 6 **and** a 7 will earn an automatic 67 points.

AR067. $(6 - 7 \div 7) \times (6 + 6) + 7$

[6]

AR068. $(66 \div (66 \div (77 - 66)) + 77) \times 77$

[7]

AR069. 666×777

[6]

AR070. $6666666666666666666666666666666666666666666666666666666666666666666$
$+ 7777777777777777777777777777777777777777777777777777777777777777777$

[7]

AR071. Which number is larger: 6^7 or 7^6?

[6]

AR072. Find $|\, 6^7 - 7^6 \,|$.

[7]

AR073. Sam's office number is MCS3015. Evaluate

$$\left(\frac{30 + 15}{3015} \times \frac{67}{6 + 7} \right).$$

[6]

AR074. Obviously 67 (base 10) = 10 (base 67). But what is 67 (base 67) in base 10?

[7]

AR075. $6! \times 7!$

[6]

AR076. $6^6 + 7^7$

[7]

AR077. What colour might have the CMYK code $\{6, 67, 67, 7\}$?

⟦*Note*: Obviously, no internet tools are allowed.⟧

[67]

Two-factor authentication... complete?

For the following questions, determine whether or not the numbers are prime. If they are not prime, provide their complete prime factor decomposition.

AR078. 67

[2]

AR079. 127

[2]

AR080. 4221

[3]

AR081. 899

[3]

AR082. 11

[2]

AR083. 101

[3]

AR084. 1001

[5]

AR085. 100000001

[13]

AR086. 314159

[7]

AR087. 117649

[7]

AR088. 117643

[7]

AR089. 33333331

[10]

AR090. 333333331

[10]

THE PERFECT PAIR

♬ *You ought to know that I think we're one and the same* ♬

It is well-known that each Mersenne prime (primes of the form $2^p - 1$, where p is also prime) has an associated perfect number. For each Mersenne number, determine whether or not it is prime. If it is, find its associated perfect number. If it isn't, provide its complete prime factor decomposition.

When faced with a number, determine whether or not it is perfect. If it is, find its associated Mersenne prime power p. If it isn't, well, move on I guess?

SNEAK PEAK: THE EUCLID-EULER THEOREM

An even number is perfect **if an only if** it is of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime (see SP006).

AR091. $2^2 - 1$

[7]

AR092. 496

[7]

AR093. 528

[7]

AR094. $2^{13} - 1$

[7]

AR095. 8128

[8]

AR096. $2^{14} - 1$

[8]

AR097. 1952

[8]

AR098. 945

[10]

AR099. $2^{23} - 1$

[10]

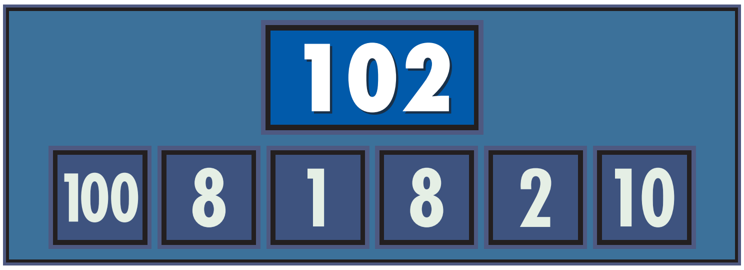
AR100. $2^{136279841} - 1$

[15]

The final countdown

For each of the following questions, use the given number cards **at most once** to obtain the desired number. You may only use basic arithmetic operations: $+, -, \times, \div$ and as many parentheses as you like. Each one should only take you 30 seconds, right?

AR101.



[1]

AR102.

[3]

AR103.

[5]

AR104.

[7]

AR105.

[7]

AR106.

[10]

AR107.

[10]

AR108. Rachel Riley lowkenuinely reaches flow state

[15]

AR109.

[15]

AR110.

[15]

AR111.

[20]

AR112.

[25]

It's Chewsday, innit?

The following questions will become a lot easier to answer if one possesses knowledge of the **Doomsday** algorithm: a neat mathematical trick popularised by group theory legend John Conway that tells you, for any given date in recent history or near future, what day of the week it lands on. It's an application of modular arithmetic and goes as follows:

The last day of February, in any given year, is Doomsday. This day of the week is shared with the following sets of dates:

$$4/4,\ 6/6,\ 8/8,\ 10/10,\ 12/12$$

$$5/9,\ 9/5,\ 7/11,\ 11/7.$$

For every three years that are not leap years, 3/1 is Doomsday. For the fourth year in a set of years that is a leap year (bar those years that are multiples of 100 but including those that are multiples of 400), 4/1 is Doomsday. Oh, and π day is Doomsday.

Doomsday 1900 was Wednesday. Doomsday 2000 was Tuesday. Moving up one year in a calendar offsets the days by one, but in the case of a leap year they are offset by two after 28 February.

⟦*Note*: There are more tricks that help you calculate Doomsday for a given year, but I will leave that as an exercise for the reader!⟧

AR113. What day of the week was 28 February 1926?

[5]

AR114. What day of the week was 1 January 1970?

[5]

AR115. What day of the week will Christmas 2131 fall on?

[5]

AR116. How many Mondays are in April 2067?

[5]

AR117. What day of the week was the Declaration of Independence signed on?

[7]

AR118. Which months of the year share the same calendar?

⟦*Hint*: A distinction between leap years may need to be made.⟧

[7]

AR119. What is the shortest number of days of the week between 31 October 2024 and 13 September 2003?

[7]

AR120. What day of the week was Sam born on? *You are allowed to ask Sam one yes/no question.*

[20]

Combinatorics

In this final section of the ARITHMETIC problems, we are going to be considering COMBINATORICS! Y'know, balls and boxes, letter arrangements, the horrible realisation that you remember nothing of what Mikhail and Clare taught you about counting problems.

Expect lots of factorials, binomial choosing and, of course, counting. No answer should contain the choose function and and any factorials less than 11! must be explicitly evaluated. As Rupi Kaur (probably) said, combinatorics is simply arithmetic; you just need to think a little harder.

~ *Seuss*

AR121. How many ways are there to arrange a 1p coin, a 2p coin, a 5p coin, a 10p coin and a 20p coin?

[5]

AR122. How many ways are there to arrange a 1p coin, two 2p coins, a 5p coin and two 10p coins?

[5]

AR123. Given that I have £5.85 in coins in my pocket, what is the minimum number of coins that would be used to achieve this?

[6]

AR124. What is the maximum number of coins that could be used to obtain £5.85 in change?

[6]

AR125. How many ways can I rearrange the letters in **SKIBIDI** to form distinct (but nonsense) words?

[6]

AR126. HOW DO YOU LIKE YOUR COFFEE IN THE MORNING?

It's 8:45am, Sam's usual arrival time in the office, and he's in need of a black coffee. In his dedicated cupboard in SW3 there are three mugs, four delicious bags of coffee beans and the choice of using an Aeropress or V60 to filter it. Sam will either grind 18 g of one type of beans or two 9 g measures of two different kind of beans.

In how many ways can Sam make a coffee, given that the cup choice matters?

[7]

AR127. How many integer solutions (x_1, x_2, x_3, x_4) are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 10,$$

given that $x_i \in \mathbb{N}_0$ for all $i \in \{1, 2, 3, 4\}$?

[8]

AR128. Evaluate $\displaystyle\sum_{n=6}^{\infty} \binom{n-1}{5} 2^{-n}$.

[8]

AR129. Take for granted that the $n^{\underline{th}}$ Fibonacci number F_n is given by Binet's formula

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}},$$

where $\phi := (1+\sqrt{5})/2$ is the golden ratio. Find the sum of the first n Fibonacci numbers in terms of ϕ, where the highest power of any exponent is n.

[8]

AR130. Chalkboard Ultra's forgotten episode

Before recording their first podcast episode, Sam and Louie decide to play chess against each other to see who has to buy the complimentary tea for their first guess. While Louie is busy setting up his side of the board, Sam realises he has *absolutely no idea* how to set up a chess board. OK, he knows that the 16 pieces must lie *somewhere* along the back two rows, but has no concept of what should go where.

In how many ways can Sam arrange his side of the chessboard?

[8]

AR131. Of these arrangements, how many will have no piece in the right place?

[8]

AR132. How many integer solutions $(x_1, x_2, x_3, x_4, x_5)$ are there to the equation

$$x_1 + x_2 - x_3 - x_4 + x_5 = 2,$$

given that $x_1 \leqslant 2$, $x_2 \leqslant 3$, $x_3 \geqslant -3$, $x_4 \geqslant 1$ and $x_5 \leqslant 5$?

[10]

AR133. The mathemagician's apprentice

It is well-known that Sam dabbles in card magic *(go on, ask him to show you one, it might get you 10 points...)*. One that he's not perfected yet is the *Any Card At Any Number*, whereby a spectator picks a card and enters it back into the deck, where it is then randomly shuffled.

A second spectator then shouts a number $C \in \{1, 2, \ldots, 52\}$. On a good day, Sam would then be able to deal the first $C - 1$ cards and reveal that the spectators card is the next card, card number C.

(a) As long as the spectator's card stays in position C, Sam may shuffle all of the other cards around it randomly. How many ways can he do this?

(b) Find the final non-zero digit of this number.

[10]

AR134. Some new security measures are being put in place for phone passcodes, whereby new 6-digit passcodes are no longer allowed to have three repeated digits in a row (ie '123336' and '455598' are not allowed, but '721011' is). What is the difference between the number of old permissible passcodes and new ones?

[10]

AR135. Stairway to Heaven... aka my office

The number of steps S one must climb in order to get to the top floor of the maths building is given by the formula

$$S = S_1 + S_2 + S_3,$$

where S_i is the number of steps it takes to get from floor $i - 1$ to floor i.

If I don't happen to take the lift all the way up, I can take the stairs by climbing either one or two steps at a time. If this choice is effectively random at each step, how many different ways can I climb the stairs?

⟦*Hint*: yes, actually get up and count them. Stair breaks are important.⟧

[12]

AR136. C is for cookie

Sarah, the resident baker for the Applied maths group, has kindly supplied all attendees at the Applied maths lunch with matcha cookies with peanut butter chocolate chips (see attached picture for one of the baking trays).

They were delicious and were the perfect snack before our seminar. But as you can see, only eight cookies fit on one of her baking trays. This tells us that multiple baking trays were used to bake all 25 cookies at once—one for everyone in the applied maths group, and one for herself.

If eight is the maximum number of cookies on one tray, and Sarah has four baking trays, how many ways can Sarah divide the cookies onto the trays?

[12]

AR137. Lily, Andreo, Paraskeví, Guy, Sam, Cai, Cassia, Lewis, Chris and Patrick are invited to a potluck at Luci's place. Of this group, five will be selected to bring a main, two people will provide drinks and the rest will provide dessert. Given that Andreo will cancel (because he hates Luci?), in how many ways can the food/drink assignments be split between everyone?

[12]

AR138. Ten-letter words are constructed from the letters DAISY.

(a) How many such words are possible?

(b) How many words have exactly three As?

(c) How many words do not contain DAISY as a subword?

[15]

AR139. Consider the setup as in AR129. Find a formula in terms of ϕ for $\sum_{k=0}^{n} {F_n}^2$, where the highest power of any exponent is n.

[15]

AR140. Evaluate $\displaystyle\sum_{n=0}^{\infty}\binom{n+7}{n}3^{-n}$.

[15]

AR141. You work in a bank. A robber comes in and demands n hundred pounds in £10, £20 or £50 notes. The robber knows that the £50 note contains a print of Alan Turing. Now, being a huge fan of Alan Turing, the robber loves adding to their collection of Alan Turing prints. They even have a spreadsheet.

So that they can accurately prepare for the addition of new prints, the robber demands to know the possible number of ways you are able to give them the money. You can choose to either tell them (and get nothing from it) or keep the information to yourself (and lose 15 marks).

[0]

AR142. The Lucas numbers L_n are defined just like the Fibonacci numbers, but we start with $2, 1$ as opposed to $0, 1$:

$$F_n := \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ F_{n-1} + F_{n-2} & n > 1 \end{cases} \qquad L_n := \begin{cases} 2 & n = 0 \\ 1 & n = 1 \\ L_{n-1} + L_{n-2} & n > 1 \end{cases}$$

By considering a Binet-like formula for L_n, find $\displaystyle\lim_{n\to\infty}\frac{L_n}{F_n}$.

[15]

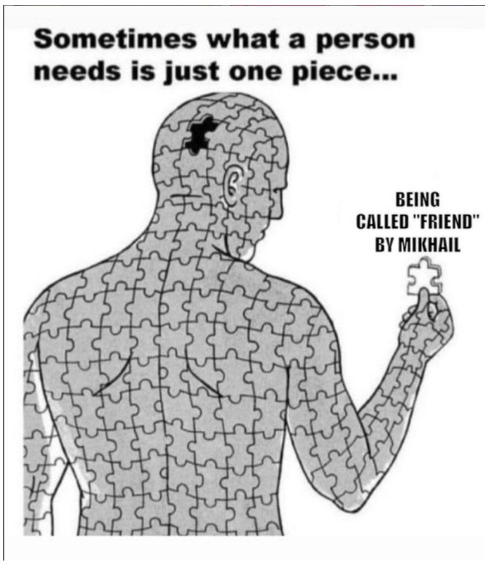
AR143. Consider the word MAGNETOHYDRODYNAMIC.

(a) How many letter arrangements of this word are there?

(b) How many of these arrangements contain the subword DECODING?

(c) How many of these arrangements **do not** contain either of the subwords TRAGIC, COMEDY?

[20]



It's fun to stay at the D-V-L-A!

For the following puzzles to do with number plates, one may choose to refer to the Comprehensive list of car registration plate areas in the appendix. Assume that any characters not representing licensing information are effectively random. The largest NN you will encounter is 75.

Yes, this section only exists because I found the UK number plate font.

Example registration plates

If a car was registered on 22 September 1974 at the DVLA office in Norwich, it may look like:

HEX 460N

If a car was registered on 13 January 1995 at the DVLA office in Birmingham, it may look like:

M177 ION

If a car was registered on 5 November 2013 at the DVLA office in Peterborough, it may look like:

AL63 BRA

AR144. Being completely obsessed with remaining the largest BNOC on Durham campus, Sam is in need of a registration plate that contains the initials SJK. Given that he needs to have it registered before he departs Durham in July 2029, how many plates are possible for him to obtain?

XXXX SJK

[10]

AR145. If a car was registered between March 2017 – February 2018, how many UK number plates of the form

XX17 XXX

XX67 XXX

are possible?

[15]

AR146. How many registration plates could Manchester and Merseyside have possibly sold between March 2004 – February 2007?

MXNN XXX

[15]

AR147. If a car has been registered since September 2001, how many UK number plates of the form

XXNN XXX

are possible?

⟦*Hint*: Note that since this system was introduced in September 2001, 01 plates did not exist; they began with 51 and so on.⟧

[15]

AR148. If instead the car used the PREFIX SYSTEM (1983-2001), how many number plates of the form

XNNN XXX

were possible?

[15]

AR149. Does this number differ from the number of possible plates using the SUFFIX SYSTEM (1963-1983)?

XXX NNNX

[15]

AR150. AND FINALLY...

Given that this converges, evaluate

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+\ldots}}}}}}$$

[25]

Sam Kay
Sam has spent so long working on this project he didn't even notice it stopped raining. In his limited off-time, he is a PhD student at Durham University working on structures in the solar wind.

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Differentiation

Polynomials

We are starting off simple, do you know your basics?

DI001. Find $\frac{dy}{dx}$ when

$$f(x) = x$$

[Marks: 1]

DI002. Find $\frac{dy}{dx}$ when

$$f(x) = x^2$$

[Marks: 1]

DI003. Find $\frac{dy}{dx}$ when

$$f(x) = 2x^4 + 9x^2$$

[Marks: 2]

DI004. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{1}{x}$$

[Marks: 2]

DI005. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{1}{x^2}$$

[Marks: 2]

DI006. Find $\frac{dy}{dx}$ when

$$f(x) = 4x^3 + 2$$

[Marks: 2]

DI007. Find $\frac{dy}{dx}$ when

$$f(x) = \sqrt{x}$$

[Marks: 2]

DI008. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \frac{1}{\sqrt{x}}$$

[Marks: 2]

DI009. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = 5x^{\frac{3}{2}}$$

[Marks: 3]

DI010. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = 2x^{\frac{5}{2}} - 7x^{\frac{1}{2}}$$

[Marks: 3]

DI011. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = (x + 1)(x - 2)$$

[Marks: 3]

DI012. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = (3x + 1)^2$$

[Marks: 3]

DI013. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = (2x - 5)^3$$

[Marks: 3]

DI014. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = (5x^3 - x)^2$$

[Marks: 3]

DI015. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \sqrt{2x + 3}$$

[Marks: 4]

DI016. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \frac{1}{\sqrt{4x - 1}}$$

[MARKS: 4]

DI017. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \frac{1}{(x + 4)^3}$$

[MARKS: 4]

DI018. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = (3x^2 + 2x - 1)^5$$

[MARKS: 5]

DI019. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \sqrt{5x^2 - 2x}$$

[MARKS: 5]

DI020. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = x(3x^2 + 1)$$

[MARKS: 5]

DI021. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = (2x - 3)(x^2 + 4)$$

[MARKS: 5]

DI022. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \frac{x}{x + 1}$$

[MARKS: 5]

DI023. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \frac{x^2}{3x - 1}$$

[MARKS: 6]

DI024. Find $\frac{dy}{dx}$ when

$$f(x) = \sin x$$

[Marks: 1]

DI025. Find $\frac{dy}{dx}$ when

$$f(x) = \tan x$$

[Marks: 2]

DI026. Find $\frac{dy}{dx}$ when

$$f(x) = 3\sin x - 2\cos x$$

[Marks: 2]

DI027. Find $\frac{dy}{dx}$ when

$$f(x) = \sin(2x)$$

[Marks: 2]

DI028. Find $\frac{dy}{dx}$ when

$$f(x) = x^2 \cos(x)$$

[Marks: 3]

DI029. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{\sin(2x)}{x^2}$$

[Marks: 4]

DI030. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{x^2}{\cos(x)}$$

[Marks: 5]

DI031. Find $\frac{dy}{dx}$ when

$$f(x) = \sin(\cos(\sin x))$$

[Marks: 5]

DI032. Find $\frac{dy}{dx}$ when

$$f(x) = \cos(3x^2 \sin x)$$

[Marks: 7]

DI033. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{\sin(x^2)}{\cos(4x)}$$

[Marks: 7]

DI034. Find $\frac{dy}{dx}$ when

$$f(x) = \log(\sin(4x^2))$$

[Marks: 7]

Exponential

Differentiate exponential functions carefully using chain and product rules.

DI035. Find $\frac{dy}{dx}$ when

$$f(x) = e^x$$

[Marks: 1]

DI036. Find $\frac{dy}{dx}$ when

$$f(x) = 5e^{2x}$$

[Marks: 2]

DI037. Find $\frac{dy}{dx}$ when

$$f(x) = e^{x^2}$$

[Marks: 3]

DI038. Find $\frac{dy}{dx}$ when

$$f(x) = xe^{3x}$$

[Marks: 4]

DI039. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{e^x}{x}$$

[Marks: 5]

DI040. Find $\frac{dy}{dx}$ when

$$f(x) = e^{\sin x}$$

[Marks: 5]

DI041. Find $\frac{dy}{dx}$ when

$$f(x) = \log(e^{2x} + 1)$$

[Marks: 5]

DI042. Find $\frac{dy}{dx}$ when

$$f(x) = e^x \cos x$$

[Marks: 5]

DI043. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{1}{e^{2x} + 3}$$

[Marks: 5]

DI044. Find $\frac{dy}{dx}$ when

$$f(x) = (x^2 + 1)e^{x^3}$$

[Marks: 6]

Hyperbolic

Recall derivatives of $\sinh x$, $\cosh x$, $\tanh x$, etc.

DI045. Find $\frac{dy}{dx}$ when

$$f(x) = \sinh x$$

[Marks: 1]

DI046. Find $\frac{dy}{dx}$ when

$$f(x) = \cosh x$$

[Marks: 1]

DI047. Find $\frac{dy}{dx}$ when

$$f(x) = \tanh x$$

[Marks: 2]

DI048. Find $\frac{dy}{dx}$ when

$$f(x) = \sinh(3x)$$

[Marks: 2]

DI049. Find $\frac{dy}{dx}$ when

$$f(x) = \cosh(x^2)$$

[Marks: 3]

DI050. Find $\frac{dy}{dx}$ when

$$f(x) = x \sinh x$$

[Marks: 3]

DI051. Find $\frac{dy}{dx}$ when

$$f(x) = \tanh(x^2 + 1)$$

[Marks: 4]

DI052. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{1}{\cosh x}$$

[Marks: 4]

DI053. Find $\frac{dy}{dx}$ when

$$f(x) = \sinh x \cosh x$$

[Marks: 4]

DI054. Find $\frac{dy}{dx}$ when

$$f(x) = \log(\cosh(2x))$$

[Marks: 5]

Inverse

Use standard derivatives of $\sin^{-1} x$, $\tan^{-1} x$, etc.

DI055. Find $\frac{dy}{dx}$ when

$$y = \sin^{-1} x$$

[Marks: 1]

DI056. Find $\frac{dy}{dx}$ when

$$y = \cos^{-1} x$$

[Marks: 2]

DI057. Find $\frac{dy}{dx}$ when

$$y = \tan^{-1} x$$

[Marks: 2]

DI058. Find $\frac{dy}{dx}$ when

$$y = \sin^{-1}(2x)$$

[Marks: 3]

DI059. Find $\frac{dy}{dx}$ when

$$y = \tan^{-1}(x^2)$$

[Marks: 4]

DI060. Find $\frac{dy}{dx}$ when

$$y = x \sin^{-1} x$$

[Marks: 5]

DI061. Find $\frac{dy}{dx}$ when

$$y = \frac{\sin^{-1} x}{x}$$

[Marks: 6]

DI062. Find $\frac{dy}{dx}$ when

$$y = \tan^{-1}\left(\frac{1}{x}\right)$$

[Marks: 6]

DI063. Find $\frac{dy}{dx}$ when

$$y = \log(\sin^{-1} x)$$

[Marks: 6]

Partial Derivatives

Find indicated first partial derivatives.

DI064. Find $\frac{\partial f}{\partial x}$ when

$$f(x, y) = x^2 y + y^3$$

[Marks: 1]

DI065. Find $\frac{\partial f}{\partial y}$ when

$$f(x,y) = x^2y + y^3$$

[Marks: 2]

DI066. Find $\frac{\partial f}{\partial x}$ when

$$f(x,y) = e^{xy}$$

[Marks: 2]

DI067. Find $\frac{\partial f}{\partial y}$ when

$$f(x,y) = \sin(xy^2)$$

[Marks: 3]

DI068. Find $\frac{\partial f}{\partial x}$ when

$$f(x,y) = \log(x^2 + y^2)$$

[Marks: 4]

DI069. Find $\frac{\partial f}{\partial y}$ when

$$f(x,y) = \frac{x}{x+y}$$

[Marks: 4]

DI070. Find $\frac{\partial f}{\partial x}$ when

$$f(x,y) = x^y$$

[Marks: 5]

DI071. Find $\frac{\partial f}{\partial y}$ when

$$f(x,y) = x^2 \cos y$$

[Marks: 5]

DI072. Find $\frac{\partial^2 f}{\partial x^2}$ when

$$f(x,y) = x^3y + e^{xy}$$

[Marks: 5]

DI073. Find $\frac{\partial^2 f}{\partial x \partial y}$ when

$$f(x, y) = x^2 y^3$$

[Marks: 5]

Implicit Differentiation

Differentiate both sides with respect to x, treating y as a function of x, and solve for $\frac{dy}{dx}$.

DI074. Given

$$x^2 + y^2 = 25,$$

find $\frac{dy}{dx}$.

[Marks: 4]

DI075. Given

$$x^3 + y^3 = 6xy,$$

find $\frac{dy}{dx}$.

[Marks: 5]

DI076. Given

$$xy + \sin y = 1,$$

find $\frac{dy}{dx}$.

[Marks: 5]

DI077. Given

$$e^y + x^2 y = 4,$$

find $\frac{dy}{dx}$.

[Marks: 6]

DI078. Given

$$\log(x + y) = xy,$$

find $\frac{dy}{dx}$.

[Marks: 6]

DI079. Given

$$x \cos y + y \sin x = 0,$$

find $\frac{dy}{dx}$.

[Marks: 6]

DI080. Given

$$\sin(xy) = x + y,$$

find $\dfrac{dy}{dx}$.

[Marks: 6]

DI081. Given

$$x^2y + y^2 = x,$$

find $\dfrac{dy}{dx}$.

[Marks: 7]

DI082. Given

$$\log y = x^2 + y,$$

find $\dfrac{dy}{dx}$.

[Marks: 7]

DI083. Given

$$x^y = y^x \qquad (x > 0,\ y > 0),$$

find $\dfrac{dy}{dx}$.

[Marks: 8]

Troll

Read the question carefully. Then read it again.

DI084. Find $\dfrac{dy}{dx}$ if

$$y = \sin^2 x + \cos^2 x.$$

[Marks: 6]

DI085. Find $\dfrac{dy}{dx}$ if

$$y = \log(1).$$

[Marks: 7]

DI086. Find $\dfrac{dy}{dx}$ if

$$y = \tan x \cdot \cot x.$$

[Marks: 9]

DI087. Find $\frac{dy}{dx}$ if

$$y = \cos(\sin^{-1} x)^2 + x^2.$$

[MARKS: 10]

REPEATED DIFFERENTIATION

Use binomial expansion, Maclaurin series, and structure of repeated derivatives carefully.

DI088. Find the 134$^{\text{th}}$ derivative of

$$f(x) = (x^2 + 1)^{67}.$$

[MARKS: 8]

DI089. Find the 67$^{\text{th}}$ derivative of

$$f(x) = \tan^{-1}(x)$$

evaluated at $x = 0$.

[MARKS: 10]

DI090.

$$\text{Let } f(x) = (x^2 + 8x + 20)e^x. \text{ Find } f^{(100)}(x).$$

[MARKS: 14]

DI091. Find the 67$^{\text{th}}$ derivative of

$$f(x) = (x + 1)\, 2^{x+1}.$$

[MARKS: 20]

MISC

Misc

DI092. Find $\frac{dy}{dx}$ when

$$f(x) = \frac{(x^2 + 1)^4}{(3x - 2)^3}.$$

[MARKS: 8]

DI093. Find $\frac{dy}{dx}$ when

$$f(x) = \log\left(\frac{x^2 + 1}{x^2 - 1}\right).$$

[MARKS: 8]

DI094. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \frac{\sin(x^2)\,\mathrm{e}^{3x}}{x^3}.$$

[Marks: 9]

DI095. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \sqrt{\frac{1+\sin x}{1-\sin x}}.$$

[Marks: 9]

DI096. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \tan^{-1}\left(\frac{x^2-1}{2x}\right).$$

[Marks: 10]

DI097. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = (\sin x)^{\cos x} \qquad (0 < x < \pi).$$

[Marks: 10]

DI098. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$f(x) = \log\Big(\log(\log(x^2+1))\Big).$$

[Marks: 12]

DI099.

Let $f(x) = \cos(x^3 - 4x^2 + 5x - 2)$. Compute $f^{(10)}(1)$.

⟦*Note*: AR045⟧

[Marks: 25]

DI100. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when

$$y = \cos^{-1}(x^2 + 67)$$

[Marks: 20]

Integration

Back to basics

Consider these first four a peace offering. A peace that will be broken if you forget the $+C$.

IN001.

$$\int x^{67}\,\mathrm{d}x$$

[Marks: 1]

IN002.

$$\int \mathrm{e}^{x}\,\mathrm{d}x$$

[Marks: 1]

IN003.

$$\int x^{-5}\,\mathrm{d}x$$

[Marks: 1]

IN004.

$$\int x\,\mathrm{d}x$$

[Marks: 1]

The calm before the storm

Don't get too cocky yet; these are all easy. You have to show me you can handle the basics before we get serious.

IN005.

$$\int_{-1}^{1} 5x^4 + x^2\,\mathrm{d}x$$

[Marks: 2]

IN006.

$$\int_{0}^{2} 3x^2 + 2x\,\mathrm{d}x$$

[Marks: 2]

IN007.

$$\int_1^3 4x^3 - 2x \, \mathrm{d}x$$

[Marks: 2]

IN008.

$$\int_0^4 2x - x^2 \, \mathrm{d}x$$

[Marks: 2]

IN009.

$$\int \sqrt{x} \, \mathrm{d}x$$

[Marks: 2]

IN010.

$$\int \frac{1}{x^2} \, \mathrm{d}x$$

[Marks: 2]

IN011.

$$\int (3x^2 - 4x + 7) \, \mathrm{d}x$$

[Marks: 2]

IN012.

$$\int \left(5 - \frac{2}{x}\right) \mathrm{d}x$$

[Marks: 2]

IN013. Are we going to start seeing trig?!

$$\int \sin x + \cos x \, \mathrm{d}x$$

[Marks: 2]

IN014.

$$\int \frac{1}{1 + x^2} \, \mathrm{d}x$$

[Marks: 2]

IN015.

$$\int \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

[Marks: 2]

IN016. Yes, yes you are

$$\int_0^{\frac{\pi}{2}} \sin x \, \mathrm{d}x$$

[Marks: 2]

IN017.

$$\int \cos(2x) - \sin(3x) \, \mathrm{d}x$$

[Marks: 3]

IN018.

$$\int \sec(x)\tan(x) - \csc^2(x) \, \mathrm{d}x$$

[Marks: 3]

IN019.

$$\int \left(2\sin(4x) + 3\cos(5x)\right) \, \mathrm{d}x$$

[Marks: 3]

IN020.

$$\int \sec^2(x) + \csc(x)\cot(x) \, \mathrm{d}x$$

[Marks: 3]

IN021.

$$\int \sinh(x) \, \mathrm{d}x$$

[Marks: 3]

IN022.

$$\int \cosh(x) \, \mathrm{d}x$$

[Marks: 3]

IN023.

$$\int \operatorname{sech}^2(x) \, \mathrm{d}x$$

[Marks: 3]

IN024. The exponent nobody asked for...

$$\int_0^1 \mathrm{e}^{\{x\}} \, \mathrm{d}x$$

⟦*Hint*: $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ indicates the floor function - the largest integer less than x.⟧

[Marks: 3]

Time to step up

You've survived the warm-ups, congrats. But now it's time to prove you actually learned something. This section turns the dial up just a little. Still nothing terrifying... just enough to make sure you're awake.

IN025. Deja vu?

$$\int \operatorname{sech}(x)\tanh(x) - \operatorname{csch}^2(x)\,\mathrm{d}x$$

[Marks: 5]

IN026.

$$\int_0^{\mathbb{1}_{\text{Riemann Hypothesis}}} \cos(\pi x)\,\mathrm{d}x, \quad \text{where } \mathbb{1} \text{ is the indicator function.}$$

[Marks: 5]

IN027.

$$\int \operatorname{csch}(x)\coth(x)\,\mathrm{d}x$$

[Marks: 5]

IN028.

$$\int_1^5 x^3 - 3x^2 + 3x\,\mathrm{d}x$$

⟦*Hint*: Integration is all about finding shortcuts. Especially when you have to do over 100 of them.⟧

[Marks: 5]

IN029.

$$\int_0^4 \frac{1}{\sqrt{|x-2|}}\,\mathrm{d}x$$

[Marks: 5]

IN030.

$$\int \frac{\mathrm{e}^{\tan x}}{1-\sin^2 x}\,\mathrm{d}x$$

[Marks: 5]

IN031.

$$\int \frac{x}{1+x^4}\,\mathrm{d}x$$

[Marks: 5]

IN032.

$$\int \frac{\arctan x}{1+x^2}\,\mathrm{d}x$$

[Marks: 5]

IN033.

$$\int_0^{\frac{\pi}{2}} \lfloor \sin x \rfloor \, dx$$

[MARKS: 5]

IN034.

$$\int_0^{e} x^2 \log(x) \, dx$$

[MARKS: 6]

IN035. Starting easy

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc(x) \cot(x) \, dx$$

[MARKS: 6]

IN036. Don't overthink it

$$\int_{-2}^{2} x^9 - 3x^7 + 2x^5 - 5x^4 + x^3 - 10x \, dx$$

[MARKS: 7]

IN037.

$$\int_5^7 \frac{1}{x^2 - 10x + 29} \, dx$$

[MARKS: 7]

IN038.

Find the value of a such that $\displaystyle\int_1^a 3x^2 - 6x + 3 \, dx = 27.$

[MARKS: 7]

IN039.

$$\int e^x e^{e^x} \, dx$$

[MARKS: 7]

IN040.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec(x) \, dx$$

[MARKS: 7]

IN041.

$$\int_0^1 \frac{1}{e^x + e^{-x}} \, dx$$

[MARKS: 7]

IN042.

$$\int_1^e (\log x)^2 \, dx$$

[Marks: 7]

Now we're getting somewhere

Since we are officially in the midfield, its time to get difficult. If the last section felt easy, don't worry, that ends here.

IN043.

$$\int_0^6 \frac{x^3 + 27}{x + 3} \, dx$$

[Marks: 8]

IN044.

$$\int_0^2 \frac{1}{(3x^2 + 4)^{\frac{3}{2}}} \, dx$$

[Marks: 8]

IN045.

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2(x)}{\sqrt{3 - \sec^2(x)}}$$

[Marks: 8]

IN046.

$$\int_2^3 \frac{1}{\sqrt{3 + 2x - x^2}} \, dx$$

[Marks: 8]

IN047.

$$\int_0^{\frac{\pi}{3}} \frac{1}{9\cos^2(x) + \sin^2(x)} \, dx$$

[Marks: 8]

IN048.

$$\int_{2024}^{2026} (x - 2025)^2 + (x - 2027)^2 \, dx$$

[Marks: 8]

IN049.

$$\int_0^2 x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 \, dx$$

[Marks: 8]

IN050. Evaluate with respect to r

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, \mathrm{d}x$$

[Marks: 8]

IN051. What did the drowning mathematician say? - Terence Tao

$$\int_{e^{e^{e}}}^{\infty} \frac{1}{x \log(x) \log(\log(x))(\log(\log(\log(x))))^{\frac{4}{3}}}\, \mathrm{d}x$$

[Marks: 8]

IN052.

$$\int_{0}^{\frac{1}{2}\log(3)} \operatorname{sech}(x)\, \mathrm{d}x$$

[Marks: 9]

IN053.

$$\int_{0}^{2} \sqrt{\frac{4-x}{x}} - \sqrt{\frac{x}{4-x}}\, \mathrm{d}x$$

[Marks: 9]

IN054.

$$\int_{0}^{\infty} \frac{2}{1+2x} - \frac{x}{1+x^2}\, \mathrm{d}x$$

[Marks: 9]

IN055.

$$\int_{0}^{\infty} \sqrt{x} e^{-x}\, \mathrm{d}x$$

[Marks: 9]

IN056.

$$\int_{0}^{1} \frac{x-1}{x^4-1}\, \mathrm{d}x$$

[Marks: 9]

IN057. Bet you wish you remembered long division

$$\int_{0}^{2} \frac{x^3 + 9x^2 + 20x + 12}{x^2 + 3x + 2}\, \mathrm{d}x$$

[Marks: 10]

IN058. How the tables have turned

$$\int_0^6 \frac{x^2+3x+2}{x^3+9x^2+20x+12}\,\mathrm{d}x$$

[Marks: 10]

IN059.

$$\int_0^{\frac{\pi}{6}} \frac{\sin(x)\cos(x)}{(\cos^2(x)-\sin^2(x))^2}\,\mathrm{d}x$$

[Marks: 10]

IN060.

$$\int_0^{\frac{\pi}{3}} \frac{\tan^3(x)}{1+\sec(x)}\,\mathrm{d}x$$

[Marks: 10]

IN061.

$$\int_{\frac{1}{2}\log(3)}^{\log(3)} \frac{1}{5\cosh(x)-4\sinh(x)}\,\mathrm{d}x$$

[Marks: 10]

IN062.

$$\int \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,\mathrm{d}x$$

[Marks: 10]

IN063.

$$\int_{-1}^{1} \frac{1}{1+\mathrm{e}^{-x}}\,\mathrm{d}x$$

[Marks: 10]

IN064. Bad day to be a mathmatician

$$\int_0^1 (\log(\log(x)))^{\frac{\log(\log(x))}{\log(\log(\log(x)))}}\,\mathrm{d}x$$

[Marks: 10]

IN065.

$$\int_1^{\infty} \frac{\arctan x}{x^2}\,\mathrm{d}x$$

[Marks: 10]

IN066.

$$\int_0^1 \sqrt{1+x^2+x^4+x^6+\ldots}\,\mathrm{d}x$$

[Marks: 10]

IN067.

$$\int_1^\infty \frac{1}{\lfloor x\rfloor\lceil x\rceil}\,\mathrm{d}x$$

[Marks: 10]

IN068. This integral is on clearance

$$\int_1^{101} \frac{\{x\}^{\lceil x\rceil}}{\{x\}^{\lfloor x\rfloor}}\,\mathrm{d}x$$

[Marks: 10]

IN069.

$$\int_0^\infty \frac{\log x}{1+x^2}\,\mathrm{d}x$$

[Marks: 10]

IN070.

$$\int \frac{\sin x}{x} + \log x \cos x\,\mathrm{d}x$$

[Marks: 10]

Shit just got serious

If you have made it this far, well done. Some of those were rude and the next ones aren't any friendlier. It's time to put your master integrator forward and give it your best shot... Good luck

You **may** need the following: $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$

IN071.

$$\int_0^{\frac{\pi}{2}} \frac{1}{(\sqrt{\sin(x)}+\sqrt{\cos(x)})^4}\,\mathrm{d}x$$

[Marks: 12]

IN072.

$$\int_{\mathrm{e}}^\infty \frac{1-\log(x)}{x^2}\,\mathrm{d}x$$

[Marks: 12]

IN073.

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\sin(2x)}\,\mathrm{d}x$$

[Marks: 12]

IN074.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,\mathrm{d}x$$

[Marks: 12]

IN075.

$$\int_0^{1} \frac{\log(x)}{1-x^2}\,\mathrm{d}x$$

[Marks: 12]

IN076. Let $f(x)=\mathrm{e}^{\cos^2(x)}$ and $g(x)=\mathrm{e}^{\sin^2(x)}$. Compute

$$\int_0^{\pi} f'(x)g'(x)\,\mathrm{d}x$$

[Marks: 12]

IN077.

$$\int_0^{\frac{\pi}{2}} \sin^6(x)+\cos^6(x)\,\mathrm{d}x$$

[Marks: 15]

IN078.

$$\int_0^{\frac{\pi}{2}} \cos(6x)\cos(7x)\,\mathrm{d}x$$

[Marks: 15]

IN079.

$$\int_0^{1} \frac{\{\cosh^2 x\}}{\{\sinh^2 x\}}\,\mathrm{d}x$$

⟦*Hint*: where $\{\cdot\}$ denotes the fractional part, $\{x\}=x-\lfloor x\rfloor$. ⟧

[Marks: 15]

IN080.

$$\int_{-\infty}^{\infty} \frac{1}{(1+\mathrm{e}^x)(1+x^2)}\,\mathrm{d}x$$

[Marks: 15]

IN081.

$$\int_{-1}^{1} \frac{1+\cos(x)}{1+3^x}\,\mathrm{d}x$$

[Marks: 15]

IN082.

$$\int_{0}^{\infty} \frac{\log x}{6767x^2+67x+6767}\,\mathrm{d}x$$

[Marks: 15]

IN083.

$$\int_{-2}^{0} \frac{x}{\sqrt{(x+2)^2+\mathrm{e}^x}}\,\mathrm{d}x$$

⟦*Hint*: Use the substitution $u = \mathrm{e}^{\frac{x}{2}}(x+2)$⟧

[Marks: 15]

IN084. The integral I can be expressed in the form $\frac{\theta^n}{n}$, where θ, n are positive real numbers. If θ represents an angle in radians, compute its value in degrees

$$I = \int_{0}^{\frac{\pi}{3}} \frac{x^{66}(\cos(x)+x\sin(x))}{\cos^{68}(x)}\,\mathrm{d}x$$

[Marks: 15]

IN085. An integral fit for a king

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^5(x)}{\sin^5(x)+\cos^5(x)}\,\mathrm{d}x$$

[Marks: 16]

IN086.

$$\int_{0}^{1} \log(1-x)\log(x)\,\mathrm{d}x$$

[Marks: 18]

IN087.

$$\int \sqrt{x\sqrt{x\sqrt{x\sqrt{x\ldots}}}}\,\mathrm{d}x$$

[Marks: 18]

IN088. Pure gold

$$\int_{0}^{\infty} \frac{1}{(1+x^{\varphi})^{\varphi}}\,\mathrm{d}x$$

[Marks: 19]

Get this guy in an integration bee

This is the make-or-break moment, the time to show what you're made of. These questions are so difficult that even WolframAlpha would need a minute; you should probably bring emotional support.

The Gamma Function

The gamma function is every integrator's second-best friend. It is an extension of the factorial over the real (and complex) numbers.

$$\text{It is defined as } \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t$$

And if $z \in \mathbb{N}$ then $\Gamma(z) = (z-1)!$

IN089.

$$\text{Solve in terms of } n \text{ and } m\text{: } \int_0^1 x^n(\log x)^m\,\mathrm{d}x$$

[Marks: 20]

IN090.

$$\int_0^1 \frac{x^3+x+1}{3x^2-3x+4}\,\mathrm{d}x$$

[Marks: 20]

IN091.

$$\int_0^2 \sqrt{x^2-x+1}-\sqrt{x^3-3x+3}\,\mathrm{d}x$$

[Marks: 20]

IN092.

$$\int_0^\infty \frac{1}{(x^2+1)^{67}}\,\mathrm{d}x \qquad \text{give your answer as a multiple of a central binomial coefficient } \binom{2n}{n}.$$

[Marks: 20]

IN093. The integral I can be expressed in the following form, where $a,\ b$ are positive integers.

$$I = \int_{\frac{1}{5}}^{\frac{1}{2}} \frac{10^x(\log(10)\log(x)-\frac{1}{x})}{\log^2 x}\,\mathrm{d}x = \frac{10^{1/a}}{\log a} - \frac{10^{1/b}}{\log b}$$

Compute a^b

[Marks: 20]

IN094. Let f be a continuous function with the property $f(x^7 + 6x^5 + 3x^3 + 1) = 9x + 5$ for all reals x. Compute the integral

$$\int_{-9}^{11} f(x)\,\mathrm{d}x$$

[Marks: 20]

IN095.

$$\int_0^{\infty} \frac{\tan^{-1}(x)}{(1+x)\sqrt{x}}\,\mathrm{d}x$$

[Marks: 25]

IN096.

$$\lim_{n\to\infty} \int_n^{n+2} \frac{\mathrm{d}x}{x^{\sin(\pi x)} + 67}$$

[Marks: 25]

IN097.

$$\int \frac{1}{x} \prod_{i=1}^{\infty} \left(1 - \tan^2\left(\frac{x}{2^i}\right)\right)\,\mathrm{d}x$$

[Marks: 25]

IN098.

$$\int_0^{\frac{\pi}{4}} \left(\frac{x}{x\sin x + \cos x}\right)^2 \mathrm{d}x$$

[Marks: 30]

IN099.

$$\text{Evaluate } \lim_{n\to\infty} I_n, \text{ where } I_0 = 1,\ I_n = \int_0^{I_{n-1}} \frac{\pi}{\pi I_{n-1} + 4x^2}\,\mathrm{d}x$$

[Marks: 67]

IN100.

$$\int_0^{2\pi} \tan(\cos x)\,\mathrm{d}x$$

[Marks: 67]

IN101.

$$\int_0^{2\pi} \sqrt{1 + \sin(2x)}\,\mathrm{d}x$$

[Marks: 67]

IN102.

$$\int \frac{\sin(1/x)}{x^3}\,\mathrm{d}x$$

[Marks: 67]

IN103. May I propose a new equation?

$$E = \int_{-AI}^{mc^2} \mathrm{d}x$$

[Marks: 67]

IN104.

$$\int_{-\infty}^{\infty} \mathrm{e}^{67+2x-x^2}\, \mathrm{d}x$$

[Marks: 67]

IN105. What the F#@*?

Let $f_n(x) = \begin{cases} 1 & \text{if } 0 < \left\{\frac{x}{n}\right\} < \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$ Evaluate $\displaystyle\int_1^{2026} \frac{1}{2}\left(1 - (-1)^{f_1(x)+f_2(x)+\ldots+f_{2026}(x)}\right) \mathrm{d}x$

[Marks: 67]

James Bramley

James is a first year studying Natural Sciences (but a physicist at heart) who simply can not stop solving integrals. When not integrating, James likes to play the saxophone and SCUBA dive.

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4

GCSE & A-Level

It's throwback time! Close your eyes and send yourself back to the good old days: classrooms, protractors and compasses that now sit unused in your desk, dates and titles, and probably one too many Kahoots.

Resist the urge to prove everything rigorously, introduce ε-δ arguments or ask whether the function is well-defined. This section exists in a much simpler universe where formula books still existed, statistics assumes data behaves itself, and integration actually means "spot the pattern and hope for the best".

You are encouraged to answer the questions quickly, nostalgically, and without referencing analysis, linear algebra, or the crushing realisation that you now overthink the easiest simultaneous equation.

And remember: once upon a time, this was really hard.

~ *Freya Jury*

LET US START FROM THE BEGINNING...GCSEs!

Radians? What are those? A *matrix* is just a film from 1999. Imaginary numbers... well they are imaginary. Let's go back to the basics.

PU001. Solve the equation $6x + 67 = 7$.

[MARKS: 1]

PU002. Solve the simultaneous equations

$$67x + 66y = 864$$
$$76x + 77y = 995$$

[MARKS: 2]

PU003. BIDMAS HAS COME EARLY!

Work out $(126 \div 3) + (-4 + 18 \div 2)^2$

[MARKS: 1]

PU004. Write $\sqrt{52} + \sqrt{117}$ in the form $a\sqrt{b}$ where a and b are integers.

[MARKS: 2]

PU005. Write $0.\dot{1}\dot{5} + 0.2\dot{2}\dot{7} + 0.1\dot{6}$ as a fraction in its simplest form.

[MARKS: 3]

PU006. A circle C has centre $(6, 7)$. A straight line L intersects circle C at points P and Q. The coordinates of P are $(-7, 13)$, and the x-coordinate of Q is 9. The line L has a negative gradient, and intersects the y-axis at the point $(0, k)$. Find the value of k.

[Marks: 4]

PU007. Just another day in MCS

Cassia is in her sixth contact hour of the day in the Scott Logic lecture theatre. She gets distracted, and instead of taking notes, somehow manages to calculate that the room has a volume of $540\ \text{m}^3$, a total surface area of $4\,380\,000\ \text{cm}^2$, and length of $12\,000$ mm. However, she then remembers that Number Theory will not teach itself, and forgets to record the room's height or width.

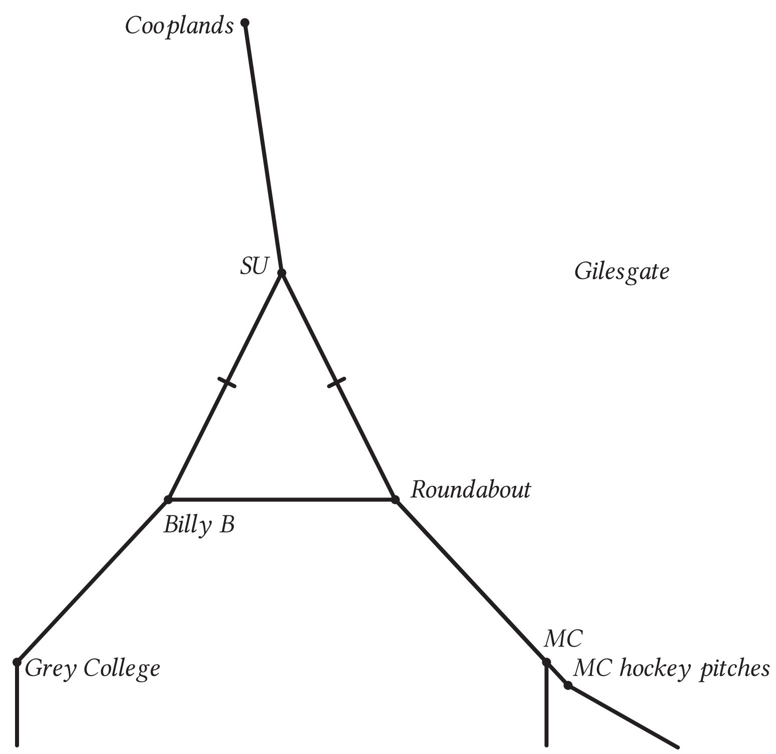
After the lecture, Freya is intrigued by Cassia's measurements and wants to recover the missing dimensions of the room. Knowing that the width is greater than the height, determine the height of the Scott Logic lecture theatre according to Cassia, giving your answer in centimetres.

[Marks: 4]

PU008. You have heard of the Bermuda triangle, but have you heard of the Gilesgate n-agon?

The diagram below shows a map of several Durham landmarks. The Billy B, the SU, and the roundabout form an isosceles triangle with the angle at the SU equal to 54°. Grey College, the Billy B, the roundabout, and MC are consecutive vertices of a regular octagon, forming its upper half.

Cooplands, the SU, the roundabout, and the MC hockey pitches are four vertices of another regular polygon, otherwise known as Gilesgate. How many sides does polygon Gilesgate have?



[Marks: 4]

PU009. Given that $x + y = 6$, let

$$3^x = \frac{3^n}{\sqrt[3]{3}} \text{ and } 3^y = (\sqrt{3})^7.$$

Work out the value of n.

[Marks: 3]

PU010. Let $X = \{4, 13, 6, 9, 10, 16, 11, 6, 9, 6\}$. Define $Y = \{\text{mean}(X), \text{mode}(X), \text{median}(X), \text{range}(X)\}$, and $Z = \{\text{mean}(Y), \text{mode}(Y), \text{median}(Y), \text{range}(Y)\}$. Work out the sum of the median, mode and range of Z.

[Marks: 3]

PU011. What is three sevenths of five sixths of half of two thirds of four fifths of seven eighths of nine tenths of two thirds of three quarters of half of 5600?

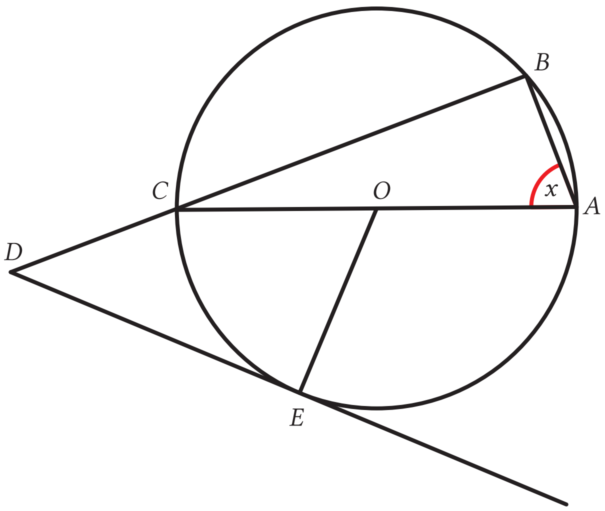
[Marks: 2]

PU012. S is a geometric sequence. The first three terms of S are $(7x - 1)$, $(7 - 4x)$ and $(6x - 21)$, where x is positive. Find the fourth term of S.

[Marks: 5]

PU013. AC is the diameter of a circle, centre O. DE is the tangent to the circle. BCD is a straight line. $AO = CD$, and angle $BAC = x$.

Express angle COD in terms of x in degrees (remember, you are a GCSE student that does not know what a radian is yet).



[Marks: 3]

PU014. Just another standard night out in Durham

I am on a mathematics society bar crawl, but got separated from the group. I know that five of my friends went to Jimmy's, and n of them went to Klute. I decide to phone two friends at random from the whole group, one after the other, to ask where they are- making sure not to call the same person twice (I am not *that* drunk). If the probability both friends I call are in Klute is $7/22$, what is the probability that one is in Jimmy's and the other is in Klute- leaving me with the difficult decision of where I would rather be?

[Marks: 3]

PU015. The functions f and g are given by

$$f(x) = \frac{7}{2x+5} \text{ and } g(x) = \frac{5x+1}{x-3}.$$

Solve the equation $fgf(x) = 1$.

[Marks: 4]

PU016. All the letters from the words MEAN, MEDIAN, and MODE are placed together in a bag. Four letters are drawn one at a time without replacement, where the order of the letters matters. What is the difference between the probability that the four letters drawn spell MODE and the probability that they spell MEAN?

[Marks: 4]

Next level unlocked: A-level Maths!

Now the real fun begins...

PU017. It is given that $(a+bx)^n = 2187 + 1701x + 567x^2 + \dots$, where a, b and n are non zero constants. Determine the values of a, b, and n.

[Marks: 6]

PU018. The equation

$$x^3 = \frac{9}{2\sqrt{x}} - 5\sqrt{x}$$

where $x > 0$, has one real positive root. Taking $x_0 = 1$ as the first approximation, apply the Newton-Raphson method once to find the exact value of x_1 in its simplest form.

[Marks: 6]

PU019. The curve C has the equation $y = f(x)$ where $x \in \mathbb{R}$. Given that

$$f(x) = \frac{3}{2}x^2 + \frac{1}{9}\sin 3x + 67,$$

the curve has a stationary point with x coordinate α, and α is small, use small angle approximations to estimate the value of α. Your answer should be in the form $a + b\sqrt{c}$ where $a, b, c \in \mathbb{Q}$.

[Marks: 5]

PU020. Find the set of values of x that satisfy the inequality

$$\left|\frac{3x}{x+3}\right| \geq 6 - 4x.$$

[Marks: 7]

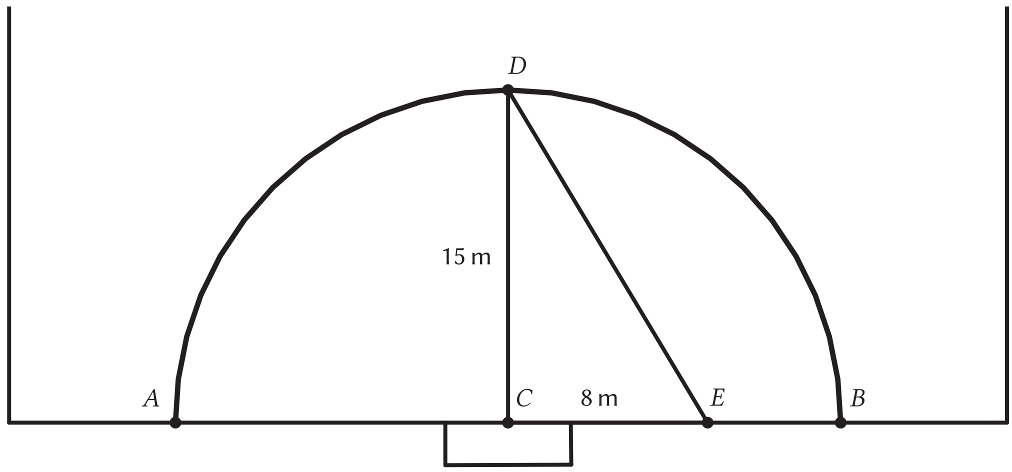
PU021. Freya likes hockey, so you must too

In hockey, a short corner is like a special penalty that gives the attacking team a chance to score. One player pushes the ball in from the back line at point E, towards a shooter at point D. Meanwhile, the defenders will run out of the goal. The centre of the goal is at point C.

The distance from the centre of the goal to the injection point is $CE = 8$ m. The defender runs directly from C to D, a distance of 15 m, and the angle DCE is a right angle. The ball is pushed from E with an initial speed of $18\,\text{m s}^{-1}$ and slows down with a constant deceleration of $2\,\text{m s}^{-2}$. The ball travels directly from E to D.

The defender starts from rest at C, but has a reaction time of $1/6$ seconds after the ball is pushed before they begin to run. By the time the ball reaches D, the defender has just reached their maximum speed of $8\,\text{m s}^{-1}$. After receiving the ball at D, the shooter takes $9/8$ seconds to align the ball to shoot. During this time, the defender continues running at a constant speed of $8\,\text{m s}^{-1}$ towards D.

At the time that the ball is finally shot at goal, how far is the defender from the ball at point D?



[Marks: 6]

PU022. Freya also likes playing the violin, so you must too

A classical orchestra's string section is arranged in a circular sector of angular width of $2\pi/3$ radians centred at the conductor. The musicians sit in rows along concentric arcs. The first row is 2 m from the conductor, and each subsequent row is 1 m further away. Each musician requires an equal arc length along their row to move their bow, and each row is fully occupied. Given that there are three rows and that the number of musicians in the third row is ten more than in the first row, find the total number of musicians and the total area of the semicircular annulus occupied by the string section. Give your answers in their exact simplified form.

[Marks: 6]

PU023. The points A and B have position vectors $5\mathbf{i} - (23/2)\mathbf{j} + 5\mathbf{k}$ and $p\mathbf{i} + 7\mathbf{j} - 11\mathbf{k}$ respectively, relative to a fixed origin O, where p is a constant. Given that triangle OAB is isosceles, find all the possible values of p.

[Marks: 6]

PU024. Solve the logarithmic equation

$$\frac{2 - 9\log_8 x}{10 - 3\log_8 x} = -(\log_8 x)^2.$$

[Marks: 4]

PU025. Curve C has the equation $y = 2x^2$. C is translated by $\begin{bmatrix} 6 \\ -7 \end{bmatrix}$ to give the equation C_1. Line L has the equation $y = x$. L is stretched by scale factor 3 parallel to the x-axis, and then translated by $\begin{bmatrix} 0 \\ -5 \end{bmatrix}$ to give the line L_1. Find the exact distance between the two intersection points of C_1 and L_1.

[Marks: 5]

PU026. Do not remind me about student loans...

A student at Durham University borrows £8,000 at the start of each academic year for a three-year degree. The loan accrues interest at an annual compound rate of 5%, with no repayments made during the course. Immediately after graduating, the student moves to Europe to work as a ski instructor, where the exchange rate is £1 = €1.10.

Calculate the total amount owed in euros at the end of the degree, giving your answer to the nearest euro.

[Marks: 5]

PU027. $\sin x = 8/17$ and x is obtuse. What is the value of $\cot 2x$?

[Marks: 5]

PU028. The equation $2k^2x^2 + (k - 6)x + 1 = 0$ has exactly two distinct real solutions, where k is a real constant. Find the range of possible values of k. Give your answer using interval notation.

[Marks: 5]

PU029. Mildly Exciting Question

At the end of Michaelmas term, 180 first year students filled in the module evaluation questionnaires (MEQs) and were asked to rate Linear Algebra, Calculus and Analysis out of five.

Four students gave all three modules the highest rating, and 112 students awarded it to Calculus. Linear Algebra received 11 fewer top ratings than Analysis, and half the students who rated Linear Algebra a top rating, also gave it to Calculus. Five more students gave the highest rating to both Calculus and Analysis than to both Calculus and Linear Algebra, and one third of students that rated Analysis five out of five, also gave that to Linear Algebra. Finally, one tenth of the students that gave Calculus and Analysis the highest rating, also gave it to Linear Algebra.

What is the probability that a randomly selected first year student that filled in the MEQs, gave none of the modules a top rating?

[Marks: 6]

PU030. Let k be a positive constant. Find the exact value of k for which

$$\int_{-3}^{1} \frac{3kx - 18}{(x+4)(x-2)}\,\mathrm{d}x = 21.$$

[Marks: 6]

PU031. The curve C has the equation

$$3^x + y^3 = 3xy - 235\cos(\pi y).$$

Find the exact value of $\mathrm{d}y/\mathrm{d}x$ at the point on C with coordinates $(6, 7)$.

[Marks: 7]

PU032. Do not worry, we have all been there

A first year maths student needs to rush up cardiac hill to the maths building for their linear algebra tutorial. The student has a mass of 70kg walks straight up cardiac hill, which is inclined at an angle θ. The length of cardiac hill measured along the slope is 80m. The student starts from rest at the bottom of the hill and walks up the slope under the action of a constant force P, applied parallel to the slope. The resistance to motion excluding gravity has a constant magnitude 40N.

After walking 20m up the hill, the student reaches a steady speed which is then maintained for the rest of the walk. The vertical height gained in walking up cardiac hill is 48m. Taking gravity as $g = 10\,\mathrm{m\,s}^2$, find the magnitude of P.

[Marks: 5]

PU033. A curve C is defined parametrically as:

$$x = \frac{2}{t+a}, \quad y = \frac{5}{t-a}, \quad t \in \mathbb{R}, \quad t \neq \pm a,$$

for some non-zero constant a. The point $P(1/3, -5/4)$ lies on C. What is the gradient of the curve at P?

[Marks: 5]

The final boss: A-level further maths

Now you are too deep into maths not to end up doing a STEM degree. You asked for more maths, and got more maths. There is no turning back now...

PU034. Solve the equation

$$z^2 - \mathrm{i}(z-2) = z - 2.$$

[Marks: 4]

PU035. Compute the total when you take the sum of all perfect squares up to 900, add the sum of all perfect cubes up to 1000, and subtract the sum of all integers from 1 to 150.

[Marks: 6]

PU036. α, β and γ are the three solutions of the cubic equation

$$x^3 - 2x^2 + x - 1 = 0, x \in \mathbb{R}.$$

Find a cubic equation with the integer coefficients whose solutions are

$$2\alpha - 1, 2\beta - 1 \text{ and } 2\gamma - 1.$$

[Marks: 7]

PU037. Solve the simultaneous equations.

$$\begin{aligned} 3x + 4y - z &= 1 \\ x - y + 2z &= -9 \\ -2x + 7y - 3z &= 26 \end{aligned}$$

[Marks: 5]

PU038. Find an expression, in terms of n, for

$$\sum_{r=2}^{n} \frac{7r+1}{r(r^2-1)}.$$

[Marks: 7]

PU039. Using de Moivre's theorem, find an expression for $\sin 5\theta$ as a function of u, where $u = \sin\theta$.

[Marks: 7]

PU040. With both direction and magnitude, oh yeah!!!

Vector wants to shoot Bob the minion with his piranha gun (oh yes!). Bob walks in a straight line from coordinates (-2, 2, -2) to (8, 0, 2). Vector unfortunately has been Freeze Ray-ed below the waist, so is stuck at the coordinate (2,-4,0). The units of distance are metres for all coordinates. However, the piranha gun was not as powerful as expected, so only has a maximum range of 5 m. Determine whether or not Vector can hit Bob with a piranha.

[Marks: 8]

PU041. Write down the matrix representing a reflection in the line $y = -x$, followed by an anticlockwise rotation by $\pi/4$ radians, followed by an enlargement by scale factor 2, in the xy-plane.

[Marks: 5]

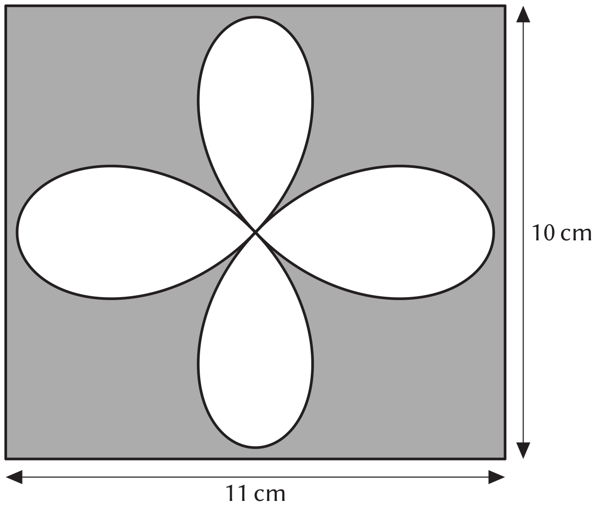
PU042. It must be your lucky day!

A leprechaun is wandering through a field when he stumbles across an impressively large four-leaf clover! The perimeter of the four-leaf clover can be modelled by the polar equation

$$r = 0.25 + 5\cos 2\theta, \qquad 0 \leq \theta < 2\pi.$$

He wants to stick the clover in the centre of his scrapbook, which has pages that are 11 cm in width and 10 cm in height. After placing the clover, he plans to decorate the remaining parts of the page.

Shown as the shaded area in the diagram below, find the exact area of the region of the page that remains visible once the clover has been added.



[Marks: 8]

PU043. Given that

$$f(x) = \log(1 + \sin 2x),$$

find the Maclaurin series for $f(x)$ up to and including the term in x^3.

[Marks: 7]

PU044. Given that

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad \text{where } 0 \leq \theta < \pi,$$

determine the values of θ for which the matrix M is singular.

[Marks: 7]

PU045. Find the three roots of the equation $z^3 = (1 - i\sqrt{3})^6(1 + i)^3$, giving your answers in the form $k\sqrt{2}e^{i\theta}$, where $-\pi < \theta < \pi$ and $k \in \mathbb{Z}$.

[Marks: 8]

PU046. The TLC cafe opens at 8:00am. The number of people in the queue at time t minutes after opening is $N(t)$, and the queue can be modelled by the differential equation

$$\frac{\mathrm{d}^2N}{\mathrm{d}t^2} + 6\frac{\mathrm{d}N}{\mathrm{d}t} + 13N = 260 + 120\cos t.$$

Initially there is no queue, and people begin to arrive at a rate of 20 people per minute. After a long time, the number of people queueing for the cafe oscillates between two fixed values- find these values.

[Marks: 8]

PU047. In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle ABC with its centre at the origin. The point A represents the complex number $9 + 3i$. If points D, E, and F are the midpoints of the sides of the triangle, find the exact area of triangle DEF.

[Marks: 9]

PU048. Find the mean value of the function $f(x) = x \arctan x$, in the interval $[0, 1]$.

[Marks: 8]

PU049. Find the exact values of x for which

$$7\cosh x - \cosh 2x = 7,$$

giving your answer in terms of natural logarithms.

[Marks: 8]

PU050. By considering the equation $z^3 = 1 + \mathrm{i}\sqrt{3}$, find the exact value of

$$\cos\frac{\pi}{9} + \cos\frac{7\pi}{9} + \cos\frac{13\pi}{9}.$$

[Marks: 9]

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5

First-year studies

Welcome, competitors, with pencils at the ready.

You should be wide awake and your hands should be steady!

This section before you has modules galore...

Some friendly, some tricky, some asking for more.

Analysis is first... try not to hide!

You'll have ε and δ by your side.

You should then try Calculus to show Bowcock who's boss,

before gearing up for LinAlg—do you remember how to cross?

Luckily these are only first-year topics... nothing curvilineal.

The stuff that in a few years' time will all be trivial.

So begin, competitors, and ponder question one.

Good spirits to you, and may Analysis be fun!

~ *Seuss*

Analysis I

Instruction to the reader: the following is best read in an old-timey English voice.

Welcome to the realm of **Analysis**, a beautiful word, as my professor was fond of saying whenever he spoke it aloud. Here you are to solve as many questions as you are able—or as many as you are willing to attempt—while bearing the particular pains that real analysis is known to inflict. Let your arguments be as formal as time allows, but do not fear: perfect and unblemished proofs are not required. Brevity, above all, is held in high regard.

Beware, for no nonsense is allowed in this realm. Every claim must be given its due warrant, save those that are truly trivial. It is never merely "obvious" that a function grows without bound, nor that a quantity may always be made smaller than epsilon—unless, indeed, it truly is obvious. You may call upon whatever theorem best serves your purpose, unless told otherwise, yet you must ever be mindful to see that its hypotheses are met.

And know this as well: in this realm, over which I hold lordship, we write `\varepsilon`. Let there be no sight of the unsightly ϵ upon your pages when I set myself to mark them.

~ Andreo Chimal

AN001. Getting real pt. 1

Remember that the real number system, $(\mathbb{R}, +, \cdot)$, is a complete ordered set. Use the real number axioms to prove the following statements. Brevity will be rewarded.

a) For every $x \in \mathbb{R}$, $x \cdot 0 = 0$.

b) For every $x \in \mathbb{R}$, $-x = (-1) \cdot x$.

[Mearcs: 4]

AN002. Getting real pt. 2

Now, use the order axioms to prove the following statements. The results from last problem may be of use. Brevity will again be rewarded. You may assume that $|x| < y$ if and only if $-y < x < y$ and $|x| > y$ if and only if $x > y$ or $x < -y$.

a) For every $x \in \mathbb{R}$, $x^2 \geq 0$.

b) For every $x, y \in \mathbb{R}$, $|x - y| \leq |x| + |y|$.

c) For every $x, y \in \mathbb{R}$, $\max\{x, y\} = \dfrac{x + y + |x - y|}{2}$ and $\min\{x, y\} = \dfrac{x + y - |x - y|}{2}$.

[Mearcs: 6]

AN003. What do you MEAN?

Let $a, b \in \mathbb{R}$ be such that $0 < a < b$. The *arithmetic mean* of a and b is $\dfrac{a+b}{2}$, and their *geometric mean* is $\sqrt{ab}$. Show that $a < \sqrt{ab} < \dfrac{a+b}{2} < b$. You may assume that $\sqrt{x^2} = |x|$ and that the square root is an increasing function. Other than that, you can only use the field and order axioms, as well as the results from the previous two problems. Be careful to not skip steps; check your assumptions constantly.

[Mearcs: 4]

AN004. At least (upper bound) it's the greatest (lower bound)

Find the infima and suprema of the following sets (if they exist). Also, decide which sets have a maximum element or a minimum element. Full proofs are not necessary, but the arguments used to obtain the values of the suprema and infima must be formal. Do not use decimal notation in your answers. Consider that $0 \notin \mathbb{N}$, however disagreeable you may find it.

a) $S = \left\{\frac{n}{n+1} \,:\, n \in \mathbb{N}\right\}$,

b) $S = \left\{\frac{1}{n} + (-1)^n \,:\, n \in \mathbb{N}\right\}$,

c) $S = \left\{x \in \mathbb{R} \,:\, x = 0.a_1a_2a_3\cdots = \sum_{k=1}^{\infty} \frac{a_k}{10^k}\right\}$, where (a_n) is a sequence of natural numbers such that $1 \le a_n \le 9$ and $a_n \neq 6, 7, 9$,

d) $S = \left\{\frac{1}{a^p} + \frac{1}{b^q} + \frac{1}{c^r} \,:\, p, q, r \in \mathbb{N}\right\}$ for $a, b, c \in \mathbb{N}$ such that $a < b < c$.

[Mearcs: 8]

AN005. Know your limits

Let $\varepsilon > 0$. For the following functions f and points x_0 in their domain, calculate the limit $\lim_{x \to x_0} f(x)$ and determine a value of $\delta > 0$ such that the definition of limit is satisfied. Once you find the value of δ, it is not necessary for you to complete a proof.

a) $f(x) = mx + b$ at $x_0 = a$ for $m, a, b \in \mathbb{R}$,

b) $f(x) = x^4$ at $x_0 = a$ for $a > 0$,

c) $f(x) = \frac{x}{1+\sin^2 x}$ at $x_0 = 0$,

d) $f(x) = \sqrt{|x|}$ at $x_0 = 0$,

e) $f(x) = \sqrt{x}$ at $x_0 = 1$.

[Mearcs: 10]

AN006. Saved by the editorial board

When the manuscript of the first edition of Michael Spivak's book *Calculus* was already at the printers, a much simpler way of proving $\lim_{x \to a} x^2 = a^2$ and $\lim_{x \to a} x^3 = a^3$ occurred to him that didn't go through all the usual steps. Suppose we wish to prove that $\lim_{x \to a} x^2 = a^2$, with $a > 0$. Given $\varepsilon > 0$, we simply take $\delta = \min\left(\sqrt{a^2+\varepsilon} - a, a - \sqrt{a^2 - \varepsilon}\right)$; then $|x - a| < \delta$ implies

$$\sqrt{a^2 - \varepsilon} < x < \sqrt{a^2 + \varepsilon},$$

so that

$$a^2 - \varepsilon < x^2 < a^2 + \varepsilon,$$

or

$$|x^2 - a^2| < \varepsilon.$$

Fortunately, he was not in time to introduce these changes because this "proof" is completely false. Where is the mistake? How can you fix it?

[Mearcs: 5]

AN007. Let $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be real-valued functions. We denote the set of discontinuities of f by $\Lambda(f)$. For the following inclusion relations, find examples that prove the inclusions are actually proper.

a) $\Lambda(f+g) \subseteq \Lambda(f) \cup \Lambda(g)$.

b) $\Lambda(cf) \subseteq \Lambda(f)$, with $c \in \mathbb{R}$.

c) $\Lambda(fg) \subseteq \Lambda(f) \cup \Lambda(g)$.

[Mearcs: 3]

AN008. Discuss the continuity of the function $f : \mathbb{Q} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if } x^2 < 2 \\ 1, & \text{if } x^2 > 2. \end{cases}$$

[Mearcs: 4]

AN009. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$|f(x) - f(y)| < (x-y)^2$$

for all $x, y \in \mathbb{R}$. Prove that f is constant.

[Mearcs: 7]

AN010. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing positive function such that

$$\lim_{x\to\infty} \frac{f(7x)}{f(x)} = 1.$$

Calculate

$$\lim_{x\to\infty} \frac{f(6x)}{f(x)}.$$

[Mearcs: 4]

AN011. Excuse me, waiter. There is a bump in my function...

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(t) = \begin{cases} \exp(-1/t), & \text{if } t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let $0 < r_1 < r_2$ and define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

Show that $h \in \mathscr{C}^\infty(\mathbb{R})$, ie it is infinitely differentiable, and that $h(t) \equiv 1$ for t such that $|t| \le r_1$ and $h(t) \equiv 0$ for $|t| \ge r_2$. We call these kinds of functions *bump functions*.

[Mearcs: 10]

AN012. Construct a function that is exactly $\mathscr{C}^6(\mathbb{R})$ but not $\mathscr{C}^7(\mathbb{R})$. That is, a function with continuous 6th derivative but no continuous 7th one. Write its correspondence rule explicitly. Can you generalise this to a function that is $\mathscr{C}^k(\mathbb{R})$ but not $\mathscr{C}^{k+1}(\mathbb{R})$ for $k \geq 1$?

[Mearcs: 4]

About integrability...

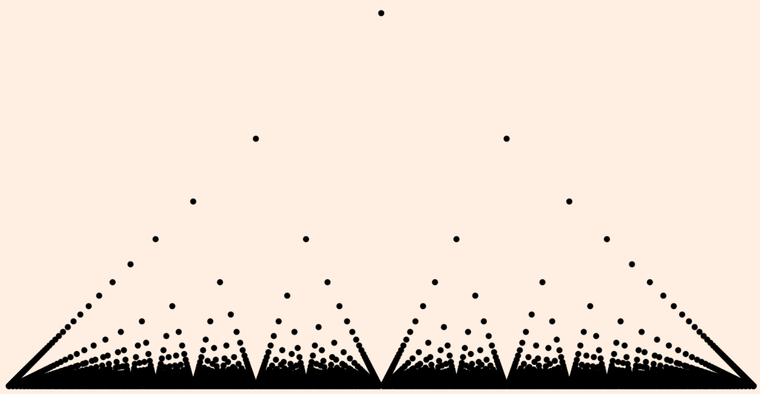
Unless stated otherwise, when the following questions mention "integrability" they refer to Riemann or Darboux integrability, which you can assume are equivalent. When asked about a function being integrable, an explicit proof is not necessary, but your arguments must show beyond a reasonable doubt that the given function is integrable. A well-explained diagram can count as an argument.

Thomae function

The *Thomae function* is a commonly used function in Analysis named after Carl Johannes Thomae, a German mathematician. Due to its helpfulness, it is known by multiple names, including —but not limited to— the popcorn function, the raindrop function, the countable cloud function and the Stars over Babylon. It is the function $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ for an irreducible fraction } p/q. \end{cases}$$

The following is a minimalistic plot of the Thomae function, it should be explanatory of the different names for the function. You can draw the axes yourselves for ε moinks.



AN013. Discuss the continuity of the Thomae function.

⟦*Hint*: It is not everywhere discontinuous.⟧

[Mearcs: 19]

AN014. Is the Thomae function integrable? If it is, find $\int_0^1 f(x)\,\mathrm{d}x$, where f is the Thomae function.

[Mearcs: 19]

The Cantor set

The *Cantor set* $\mathscr{C}$ is a set whose construction is frequent in Analysis; it is named after the Russian-German set theorist Georg Cantor. This set is constructed recursively as an intersection of a decreasing sequence of closed sets $\mathscr{C}_n \subset [0,1]$.

Start with $\mathscr{C}_0 = [0,1]$. At step $n = 1$, remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ to obtain

$$\mathscr{C}_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

For $n = 2$, remove the open middle third of each remaining interval, giving

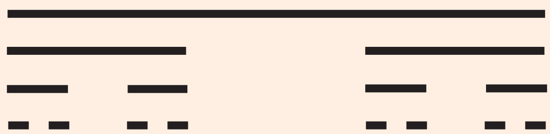
$$\mathscr{C}_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Proceed inductively: $\mathscr{C}_n$ consists of 2^n disjoint closed intervals, each of length 3^{-n}, obtained by removing the open middle third of every interval in $\mathscr{C}_{n-1}$.

The Cantor set is then defined as

$$\mathscr{C} = \bigcap_{n=0}^{\infty} \mathscr{C}_n.$$

The following is a minimalistic plot of the first four iterations of the Cantor set construction.



AN015. Let $A \subseteq \mathbb{R}^n$ be a non-empty set. The *characteristic function* of A is the function $\chi_A : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

Is the characteristic function of the Cantor set, $\chi_{\mathscr{C}}$, integrable? If it is, find $\int_0^1 \chi_{\mathscr{C}}(x)\,\mathrm{d}x$.

⟦*Hint*: Fist consider $\chi_{\mathscr{C}_n}$ and define a partition such that the upper sum of $\chi_{\mathscr{C}_n}$ subordinated to the partition equals $(2/3)^n$.⟧

[Mearcs: 12]

AN016. Find $\Lambda(\chi_{\mathscr{C}})$, the set of discontinuities of the Cantor set.

[Mearcs: 15]

AN017. Let f and g be real valued functions. Decide if the following statements are **true** or **false**. If true, provide a simple argument to justify them; if false, provide a counterexample.

a) If a function is integrable, then any upper sum is equal to any lower sum.

b) If $f^2 + g$ and g are integrable, then f is integrable.

c) If f is integrable and $\int_a^b f(x)\,\mathrm{d}x = 0$, then there exists a point $x_0 \in [a,b]$ such that $f(x) = 0$.

d) If f is not integrable, then $|f|$ is not integrable.

e) If f has infinitely many discontinuities, then it is not integrable.

f) If $\Lambda(f)$ can be regarded as a non-convergent sequence, then f is not integrable.

g) If f is discontinuous on a dense subset of its domain, then f is not integrable.

h) If $\Lambda(f)$ is uncountable, then f is not integrable.

i) If f is discontinuous at every term of the sequence (x_n) and $x_n \to x_0$ with $x_0 \in [a,b]$, then f is discontinuous at x_0.

j) If g does not have a primitive (ie a function F such that $F' = g$), then it is not continuous.

⟦*Hint*: The Thomae function and the characteristic function of the Cantor set may be useful for some of these.⟧

[Mearcs: 20]

AN018. Let $f : [a,b] \to \mathbb{R}$ be such that $f(a) = 0 = f(b)$. Suppose that f' exists, it is continuous in $[a,b]$ and $\displaystyle\int_a^b f^2(x)\,\mathrm{d}x = 1$. First, calculate

$$\int_a^b x f(x) f'(x)\,\mathrm{d}x,$$

and then use that to show that

$$\frac{1}{4} \leq \left(\int_a^b (f'(x))^2\,\mathrm{d}x\right)\left(\int_a^b x^2 f^2(x)\,\mathrm{d}x\right).$$

[Mearcs: 8]

AN019. Let $g : [a,b] \subset \mathbb{R} \to [c,d] \subset \mathbb{R}$ and $f : [c,d] \subset \mathbb{R} \to \mathbb{R}$ be two integrable functions. Consider the composition $h := f \circ g : [a,b] \subset \mathbb{R} \to \mathbb{R}$. Show an example in which h is integrable and one in which it is not.

[Mearcs: 10]

AN020. Let $\mathscr{R} = [0,2] \times [0,1]$ and $f : \mathscr{R} \to \mathbb{R}$ be given by

$$f(x,y) = \frac{xy(x^2 - y^2)}{(x^2 + y^2)^3}.$$

Using a simple u-substitution, calculate

$$\int_0^1 \int_0^2 f(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{and} \qquad \int_0^2 \int_0^1 f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

What is going on? Is Fubini's theorem wrong?

[Mearcs: 8]

Metric spaces

A *metric space* (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}$ called the *metric*, such that for all $x, y, z \in X$ the following properties are satisfied:

1. **Positivity:** $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$;
2. **Symmetry:** $d(x, y) = d(y, x)$;
3. **Triangle Inequality:** $d(x, y) \leq d(x, z) + d(z, y)$.

Sequence convergence can be written in terms of any metric. If (x_n) is a sequence in X, it converges to $x_0 \in X$ in the metric d if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) < \varepsilon$ is satisfied. This is usually denoted by $x_n \xrightarrow{d} x_0$.

AN021. Consider $X = \mathbb{R}^n$. Let d be given by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

Is (X, d) a metric space? Characterise all the convergent sequences in X.

[Mearcs: 5]

AN022. Still considering $X = \mathbb{R}^n$, now let d be given by

$$d(x, y) = \sqrt{\|x - y\|}.$$

Is (X, d) a metric space? Denoting by d_2 the usual euclidean distance, $d_2(x, y) = \|x - y\|$, is it true that a sequence (x_n) converges in (X, d) if and only if it converges in (X, d_2)? Justify why or provide a counterexample.

[Mearcs: 8]

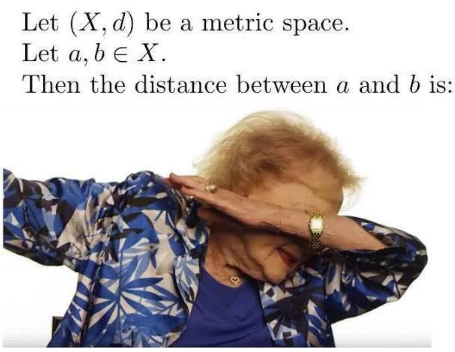
AN023. Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a function such that, for all $x, y, z \in X$,

i. $d(x, y) = 0 \iff x = y$, and

ii. $d(x, z) \leq d(x, y) + d(y, z)$.

Is X a metric space? If not, add a single property to the list to make X a metric space and provide an example that satisfies properties i and ii and is not a metric space.

[Mearcs: 6]



Topology on metric spaces

If (X, d) is a metric space, given $r > 0$ and $x \in X$ we define the *open ball centred around x with radius r* as the set

$$B_r(x) := \{y \in X : d(x, y) < r\}.$$

Given a set $A \subseteq X$, we define:

1) The *interior* of A, $\text{int}(A) := \{x \in X : \exists r > 0,\ B_r(x) \subseteq A\}$. If $x \in \text{int}(A)$, it is called an *interior point.*
2) The *closure* of A, $\text{cl}(A) := \{x \in X : \forall r > 0,\ B_r(x) \cap A \neq \emptyset\}$. If $x \in \text{cl}(A)$, it is called an *adherent* or *closure point.*
3) The *derived set* of A, $A' := \{x \in X : \forall r > 0,\ (B_r(x) \setminus \{x\}) \cap A \neq \emptyset\}$. If $x \in A'$, it is called a *limit* or *accumulation point.*
4) The *exterior* of A, $\text{ext}(A) := \{x \in X : \exists r > 0,\ B_r(x) \subseteq X \setminus A\}$. If $x \in \text{ext}(A)$, it is called an *exterior point.*
5) The *boundary* of A, $\partial A := \{x \in X : \forall r > 0,\ B_r(x) \cap A \neq \emptyset \text{ and } B_r(x) \cap (X \setminus A) \neq \emptyset\}$. If $x \in \partial A$, it is called a *boundary point.*

Additionally, we say A is *open* if $A = \text{int}\, A$ and *closed* if $A = \text{cl}\, A$.

The set of all open sets determined by d is called the *topology induced by d*, which we denote by τ_d or, simply, τ. It is easy to verify that it satisfies the following properties:

i. $\emptyset, X \in \tau$;

ii. if $\{U_\alpha\}_{\alpha \in \mathscr{A}} \subseteq \tau$ for an arbitrary set of indices $\mathscr{A}$, then $\bigcup_{\alpha \in \mathscr{A}} U_\alpha \in \tau$, and

iii. if $U, V \in \tau$, then $U \cap V \in \tau$.

AN024. Decide whether the following statements are true or false. If true, provide a simple argument to justify them; if false, provide a counterexample. Let $U, V \subseteq X$ be subsets of a metric space.

a) If U is not closed, it must be open.

b) U may be both open and closed.

c) U may be not open and not closed.

d) If $U, V \subseteq X$, then $\text{cl}\,(U \cap V) = \text{cl}\, U \cap \text{cl}\, V$.

e) U is closed if and only if $U' \subseteq U$.

f) $\text{ext}\, U = X \setminus (\text{int}\, U)$.

g) $\partial(\text{int}\, U) = \partial U$ and $\partial(\text{cl}\, U) = \partial U$.

[Mearcs: 16]

AN025. Find a metric for $\mathbb{R}^n$ such that every subset of $\mathbb{R}^n$ is both open and closed.

[Mearcs: 5]

AN026. Given a set A and a natural n, let us denote $A^n := \underbrace{A \times \cdots \times A}_{n \text{ times}}$. Consider the set $S = \mathbb{Q}^n \cap [0,1]^n \subset \mathbb{R}^n$. The euclidean metric is the usual metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $d(x,y) = \|x - y\|$. Find the interior, closure, derived set, exterior and boundary of A in the euclidean topology $(\mathbb{R}, \tau_d)$. Is A open? Is it closed?

[Mearcs: 5]

AN027. Let $I_k = \left(-\frac{1}{k}, 1 + \frac{1}{k}\right)$ for $k \in \mathbb{N}$. Find $\bigcap_{k \in \mathbb{N}} I_k$ and use it to prove that the intersection of a countable number of open sets need not be open. Is the same true for a finite intersection?

[Mearcs: 3]

AN028. Is the union of a countable number of closed sets necessarily closed? If your answer is negative, show a counterexample.

[Mearcs: 3]

AN029. Can every countable set be expressed as a countable union of closed sets?

[Mearcs: 3]

AN030. A set S is called *perfect* if every point of S is a limit point, ie $S' = S$. Is the Cantor set perfect? Is it closed? Explain.

[Mearcs: 5]

AN031. A real number x is called an *almost upper bound* of $A \subseteq \mathbb{R}$ if there exist only finitely many numbers $y \in A$ such that $y \geq x$. Similarly, you can define an *almost lower bound*.

a) Find all the almost lower bounds and the almost upper bounds of the sets in AN004.

b) Suppose that A is an infinite bounded set. The set B of all almost upper bounds of A is non-empty and bounded below. Why? It follows that $\inf B$ exists; this number is called the *limit superior* of A, and is denoted by $\limsup A$ or $\overline{\lim} A$. Define $\liminf A$, or $\underline{\lim} A$, analogously.

c) Find lim sup and lim inf for each of the sets in AN004.

[Mearcs: 6]

AN032. The limit superior and limit inferior of a sequence can also be defined. Given a bounded sequence (x_n), note that the sequence $(\sup\{x_k : k \geq n\})_{n \in \mathbb{N}}$ is decreasing and bounded below. Why? Analogously, $(\inf\{x_k : k \geq n\})_{n \in \mathbb{N}}$ is increasing and bounded above. This means that they both converge, so we can define

$$\limsup x_n := \lim_{n \to \infty} \sup_{k \geq n} x_k = \inf_n \sup_{k \geq n} x_k \quad \text{and} \quad \liminf x_n := \lim_{n \to \infty} \inf_{k \geq n} x_k = \sup_n \inf_{k \geq n} x_k.$$

Find lim sup (or $\overline{\lim}$) and lim inf (or $\underline{\lim}$) for the sequences

a) $\left(\frac{(-1)^{n+1}}{n}\right)$,

b) $\left((-1)^{n+1}\frac{n+1}{n+2}\right)$.

[Mearcs: 5]

AN033. Calculate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(3k-2)^{66}}{3^{66}n^{67}}$$

without using the binomial theorem (not that anybody would be crazy enough to do it that way).

[[*Hint*: Think Riemann.]]

[Mearcs: 6]

AN034. Discuss the convergence and the uniform convergence of the series $\sum_{n=1}^{\infty} f_n$, where $f_n(x)$ is given by

a) $(x^2+n^2)^{-1}$,

b) $\sin(x/n^2)$,

c) $(-1)^n(n+x)^{-1}, x \geq 0$.

[Mearcs: 18]

AN035. Too geometric for its own good

Let $a, b, c \in \mathbb{R}$ be such that none of them are zero or a negative integer. The hypergeometric series is defined by

$$\frac{ab}{1!c}+\frac{a(a+1)b(b+1)}{2!c(c+1)}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}+\cdots.$$

Discuss the convergence and absolute convergence of the hypergeometric series.

[[*Hint*: Remember Raabe's test? You know, the one that nobody bothers to learn.]]

[Mearcs: 15]

AN036. Consider the series

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\cdots,$$

where the signs come in groups of two. Does it converge? Does it converge absolutely?

[Mearcs: 8]

AN037. Did he just say "Euler macaroni"?

For $n \in \mathbb{N}$, let c_n be defined by

$$c_n=\sum_{k=1}^{n}\frac{1}{k}-\log n,$$

where log is the natural logarithm. Show that (c_n) is a monotone decreasing sequence of positive numbers. The limit of this sequence is called the Euler-Mascheroni constant, denoted by γ. It has an approximate value of 0.577...

[[*Hint*: To analyse —*beautiful word*— the sequence (c_n), the integral definition of the natural log may come in handy.]]

[Mearcs: 14]

AN038. The saga continues...

Show, without using Taylor expansions, that if we put $b_n = 1 - 1/2 + 1/3 - \cdots - 1/2n$, then the sequence (b_n) converges to $\log 2$.

⟦*Hint*: First show that $b_n = c_{2n} - c_n + \log 2$, where c_n is the sequence defined in the previous problem.⟧

[Mearcs: 10]

AN039. And the gift keeps on giving...

Show that the series

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + + - \cdots$$

diverges (the sign pattern is two pluses and a minus subsequently alternating).

⟦*Hint*: Draw some inspiration from the previous problem.⟧

[Mearcs: 12]

AN040. Calculate the value of the following series and simplify them as much as possible:

a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!3^k}$.

b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k 4^k \pi^{2k}}{(2k)!}$.

c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(e-1)^k}$.

⟦*Hint*: Think Taylor.⟧

[Mearcs: 9]

Andreo Chimal
Andreo is a PhD student at Durham University working on Geophysical Fluid Dynamics and windfarm modelling, though he is a closeted geometer. He enjoys linguistics, salsa dancing and talking about himself in the third person.

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Calculus I

> That's right Sam, I'm finally typing up...
>
> ~ *Luci Mullen*

CA001. Let

$$I_n = \int_1^2 \left(1 - \frac{1}{x}\right)^n \mathrm{d}x.$$

Use integration by parts to show that

$$I_{n+1} = \alpha I_n + 2^\beta \gamma,$$

where α, β, γ are all terms depending only on n that you should find.

[Marks: 18]

CA002. Let $f(t)$, $\frac{df}{dt}$, $y(x)$, $z(x)$ be continuous functions. The fundamental theorem of calculus states that for some $a \in \mathbb{R}$,

$$\frac{d}{dx}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

1. Let

   $$F_1(x) = \int_a^{y(x)} f(t)\,\mathrm{d}t.$$

   Write an expression for $F_1'(x)$ in its simplest form.

2. Let

   $$F_2(x) = \int_{z(x)}^{y(x)} f(t)\,\mathrm{d}t.$$

   Write an expression for $F_2'(x)$ in its simplest form.

3. Let $f(x,t)$ and $\frac{\partial f}{\partial x}(x,t)$ be continuous functions. The definite integral of a partial derivative states that for some $a, b \in \mathbb{R}$,

   $$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(x,t)\,\mathrm{d}t = \int_a^b \frac{\partial}{\partial x} f(x,t)\,\mathrm{d}t.$$

   Let

   $$F_3(x) = \int_{z(x)}^{y(x)} f(x,t)\,\mathrm{d}t.$$

   Write an expression for $F_3'(x)$ in its simplest form.

[Marks: 15]

CA003. Find the derivative with respect to x of the integral

$$I(x) = \int_x^{x^2} \frac{\sin(xt)}{t}\,\mathrm{d}t.$$

[Marks: 8]

CA004. Solve the differential equation

$$\frac{d^2y}{dx^2} = \left(1+3x^2\right)\left(\frac{dy}{dx}\right)^2; \qquad y(1)=0, \quad y'(1)=-\frac{1}{2}$$

[Marks: 7]

CA005. Solve the differential equation

$$\left(1+x^2\right)\frac{dy}{dx} = x^4(1-x)^4, \qquad y(0)=0.$$

Hence, find the value of $y(1)$.

[Marks: 9]

CA006. Solve the differential equation

$$\frac{dy}{dx} + \tan(x)y = \sec^4(x)y^3.$$

[Marks: 7]

CA007. Solve the differential equation

$$\frac{dy}{dx} = \frac{\sin^{-1}(x)}{y^2\sqrt{1-x^2}}; \qquad y(0)=0.$$

[Marks: 13]

CA008. Solve the differential equation

$$\frac{dy}{dx} - y\tan(x) = 1; \qquad y(0)=1.$$

[Marks: 5]

CA009. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}.$$

[Marks: 12]

CA010. Solve the differential equation

$$x\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

.

[Marks: 12]

CA011. Got the 'schitz

Consider the differential equation

$$\frac{dy}{dx} = \sqrt{1+y}; \qquad y(0) = 0.$$

1. Solve the differential equation to find a solution for $y(x)$. Sketch the graph of the solution.
2. Consider your value of $y'(x)$ for $x < -2$, both from your solution and from the differential equation. What do you notice?
3. What is going wrong here?
4. Write a solution which is continuous and smooth for all $y \in \mathbb{R}$.

[Marks: 18]

CA012. Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}; \qquad y(0) = 4, \quad y(-\log(3)) = 6 - 3\log(3).$$

[Marks: 14]

CA013. Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = \sin(2x); \qquad y(0) = 0, \quad y'(0) = 1.$$

[Marks: 11]

CA014. Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{x}\cos(2x); \qquad y(0) = 1, \quad y'(0) = 0y.$$

[Marks: 10]

CA015. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{1}{x}\frac{dy}{dx} + \frac{6}{x^2} = \frac{1}{x^6}.$$

[Marks: 7]

CA016. Find the implicit solution and the domain of $y(x)$ where

$$\left(x + y^3\right)\frac{dy}{dx} = y; \quad y(1) = 2.$$

[Marks: 13]

CA017. Solve

$$\left(y - \frac{2y}{x^2 - y^2}\right) \mathrm{d}x + \left(x + \frac{2x}{x^2 - y^2}\right) \mathrm{d}y = 0.$$

Answers may be left in implicit form.

[Marks: 9]

CA018. The Guy Carparker Parkrun special

Let $\alpha(x), \beta(x), \gamma(x) \in C([a,b])$ and let $y \in C^2((a,b) \cap C([a,b])$ denote a solution of the ordinary differential equation given by

$$\begin{cases} \alpha y''(x) + \beta y'(x) + \gamma y(x) = 0, & \text{on } (a,b); \\ y = 0, & \text{on } \{a\} \cup \{b\}. \end{cases}$$

If $\alpha(x) > 0$ for all $x \in [a,b]$ and $\gamma(x) < 0$ for all $x \in [a,b]$, then prove that $y(x) = 0$ for all $x \in [a,b]$.

[Marks: 9]

CA019. That's a mean function you've got there

Assume $f(x)$ is a function with a continuous derivative and satisfies $f'(1) = 0$. Show that there exists some $c \in \mathbb{R}$ such that

$$c^2 f'(c) + 2cf(c) = 0.$$

⟦*Hint*: MVT⟧

[Marks: 8]

CA020. Remy who?

Luci is horrified to discover a population of rats confined to the second floor of the maths department. Luci poisons the second floor of the maths department with a toxin dangerous only to rats. However, Sam begins to panic since he, himself, is a rat, and needs to escape as quickly as possible. Modelling the second floor as the xy–plane, Sam is standing at $(1,2)$ (the men's bathrooms), and the concentration of gas is given by the function $f(x,y) = 2x^2y - 3x^3$. In which direction away from the men's bathrooms should Sam initially move in order to lower his exposure to the toxin as rapidly as possible and thus ensure that Le Maths continues as normal? Give your answer in the form of a unit vector.

[Marks: 4]

CA021. Simplify

$$\nabla \times \boldsymbol{a}\,(\nabla \cdot \boldsymbol{a}) + \boldsymbol{a} \times (\nabla \times \nabla \times \boldsymbol{a}) + \boldsymbol{a} \times \nabla^2 \boldsymbol{a}.$$

[Marks: 5]

CA022. Evaluate the Laplacian of the function

$$f(x,y,z) = \frac{x^2 z}{x^2 + y^2 + z^2}$$

directly in Cartesian coordinates.

[Marks: 3]

CA023. Find and classify all the critical points of

$$f(x, y) = \sin(x)^2 + \sin(y)^2 - \cos(x)^2 \cos(y)^2.$$

[Marks: 6]

CA024. Let

$$f(x, y) = x^3 + x^2 + 2\alpha xy + y^2 + 2\alpha x + 2y,$$

where α is a positive constant. Find and classify all the critical points of f(x, y) when

1. $\alpha > 1$,
2. $0 < \alpha < 1$.

[Marks: 5]

CA025. I HEART PANCAKES

At Luci's annual pancake day celebrations, Sam manages to get his hands on the pancake batter, which costs approximately 1.5p per 1cm^3. He quickly manages to get some of the batter in the frying pan before Luci drags the bowl away from him. The ensuing brief squabble means that the the mix isn't properly distributed in the pan, and Luci notices that it looks like a cardioid. If we model the edge of the pancake with the polar equation

$$r = a(1 + \cos(\theta)), \quad 0 \leq \theta < 2\pi,$$

where $a = 5\text{cm}$.
If we model the pancake as a cardioid–based cylinder which is 1mm thick, how much of Luci's PhD stipend was lost on Sam's ugly pancake?

[Marks: 6]

CA026. In this question we use spherical polar coordinates. For any body which is symmetrical about the polar axis, the element of volume for the body is $dV = 2\pi r^2 \sin(\theta) dr d\theta$, and the element of surface area is $2\pi r \sin(\theta)\sqrt{(dr)^2 + r^2(d\theta)^2}$. We now consider a particular surface, defined by $r = 2a\cos(\theta)$, where a is constant and $0 \leq \theta \leq \pi/2$. Find the total surface area of the body, and the volume that it encloses. Identify the surface.

[Marks: 6]

CA027. Find the value of

$$\int\int\int x^2 y \, dx \, dy \, dz$$

over the volume of a tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

[Marks: 4]

CA028. Evaluate

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} \, dx.$$

[Marks: 4]

CA029. Consider an ellipsoid defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Find the volume of the largest box which can fit inside of the ellipsoid, given that its edges must be parallel to the x-,y-,z-axes.

[Marks: 12]

CA030. Find all the critical points (x, y, z) of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints

$$x + y - z = 0, \quad \text{and} \quad \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1.$$

Determine the value of $f(x, y, z)$ at these points.

[Marks: 16]

CA031. Use the Taylor series for a function of two variables to expand the function

$$f(x) = x^4 + y^3 + 7x^2y$$

about the point $(1, 2)$.

[Marks: 2]

CA032. Evaluate

$$\lim_{x \to 0} \left(\frac{\tan(x) - \sin x}{x^3} \right)$$

[Marks: 5]

CA033. For what value of k is

$$\lim_{x \to 0} \left(\frac{-5x + kx^3 + \sin(5x)}{x^5} \right)$$

finite? Find the value of this limit.

[Marks: 3]

CA034. Find coefficient of x^n in the Taylor series of

$$\frac{8}{(4 - x)(1 + x)^2}$$

and state the set of values of x for which this expansion is valid.

[Marks: 13]

CA035. Find the first four nonzero terms in a power series expansion of the solution to the problem

$$y''(x) - \sin(x)y(x) = 0,$$

with initial values

$$y(0) = 1, \ y'(0) = 0.$$

[Marks: 11]

CA036. Let $f(x), g(x)$ be functions which are analytic at $x = 1, b$ where a, b are real numbers. Consider the operator

$$\mathscr{L} = (x-3)(x+1)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(x-1)\frac{\mathrm{d}}{\mathrm{d}x}.$$

Find values of a, b so that $\mathscr{L}$ is self-adjoint with respect to the inner product

$$(f(x), g(x)) = \int_a^b f(x)g(x)\,dx.$$

[Marks: 4]

CA037. Consider the second differential equation

$$\alpha\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \beta\frac{\mathrm{d}^2 y}{\mathrm{d}x\,\mathrm{d}y} + \gamma\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0,$$

where α, β, γ are all constants. By making substitutions of the form $\xi = x + my$, $\eta = x + ny$, where m, n are constants, determine conditions on m, n in terms of α, β, γ that ensure that, after a change of variables, the differential equation is of the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi\,\mathrm{d}\eta}.$$

[Marks: 7]

CA038. Now consider the differential equation

$$6\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - 5\frac{\mathrm{d}^2 y}{\mathrm{d}x\,\mathrm{d}y} + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$$

Use the previous question to select m, n (with $m \neq n$) so that after a change of variables the differential equation can be written in the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi\,\mathrm{d}\eta}.$$

If, on the line $y = 0$, $u = 2x + 1$ and $\partial u/\partial y = 4 - 6x$, show that it must be possible to write u in the form

$$u(x, y) = F(x + 2y) + G(x + 3y),$$

and determine the form of F and G.
Hence, write the full solution for u in terms of x and y.

[Marks: 9]

CA039. The periodic function $f(x)$ of period 2π is defined by

$$f(x) = x\sin(px) \text{ for } -\pi < x < \pi,$$

where p is a positive integer.

1. Use integration by parts to show that if m is a non-zero integer then

$$\int_0^{\pi} x\sin mx\,dx = \frac{\pi(-1)^{m+1}}{m}.$$

.

2. Show that the Fourier series for f is given by

$$f(x) \sim \frac{(-1)^{p+1}}{p} - \frac{\cos(px)}{2p} + \sum_{n=1, n\neq p}^{\infty} \frac{2p(-1)^{n+p}}{n^2 - p^2}\cos(nx).$$

⟦*Hint*: Consider whether $f(x)$ is an even or an odd function.⟧

3. Use the Fourier convergence theorem to deduce the values of

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \quad \text{and} \quad \sum_{n=3}^{\infty} \frac{(-1)^n}{n^2-4}.$$

[Marks: 19]

CA040. The periodic function $f(x)$ of period $2L$ is defined by

$$f(x) = \begin{cases} 0, & -L \le x < 0, \\ 1 - \frac{x}{L}, & 0 \le x < L. \end{cases}$$

Find the Fourier series for $f(x)$, and by considering the value of the series at appropriate values for x, deduce the values of

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

[Marks: 20]

Luci Mullen
Luci is a doctoral teaching fellow at Durham University. When they're not teaching 12 tutorials a week, they can be found arguing with the weakly nonlinear analysis of reaction–cross-diffusion equations, presenting posters, or occupying other people's offices. Outside of maths, Luci enjoys knitting, creating colourful artwork, and sports which involve descending mountains as fast as humanly possible.

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Linear algebra I

Welcome to the eras tour ... well not quite. Let's reset, welcome to the **Linear Algebra** section!

Here we've traded the friendship bracelets for basis sets, and whilst the backing singers are harmonising we'll be normalising. In this section, we aren't just looking for 'Style' by simplifying our answers as much as possible — we're also looking for Eigenspaces.

We will move through the developmental eras of a mathematician. First, there was the "Childhood Innocence Era," where x was just a number. Then came the "Matrix Chaos Era," where you realized AB doesn't actually equal BA and felt personally betrayed by the laws of arithmetic. Now, we've reached the Gram-Schmidt Era, the ultimate mathematical glow-up where we take a messy basis and force it to be orthonormal— because honestly, if your vectors aren't at right angles and unit length, are they even worth knowing?

Whether you're projecting your problems onto a vector subspace or just hoping your determinant isn't zero, remember: in this section, we don't have bad blood, we just have linearly independent differences.

~ *Cassia Pearce*

LA001. Do I spy another Six Seven?

$$\begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 0 & 6 \\ 7 & 0 \end{bmatrix} =$$

[Marks: 1]

LA002. Perhaps multiple occurrences of Six Seven...

$$\begin{bmatrix} 6 & 7 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 6 & 6 \\ 7 & 7 \end{bmatrix} =$$

[Marks: 1]

LA003. Correct that to too many Six Sevens

$$\boldsymbol{A} = \begin{bmatrix} 6 & 7 & 7 \\ 6 & 6 & 7 \end{bmatrix}$$

Write down $\boldsymbol{A}^T$.

[Marks: 1]

LA004.

$$\boldsymbol{B} = \begin{bmatrix} 67\delta_{ca} & 706\delta_{ss} \\ 76\delta_{ia} & 667 \end{bmatrix}$$

Write down $\boldsymbol{B}$ such that $\boldsymbol{B} \in M_{2,2}(\mathbb{Z})$.

[Marks: 1]

LA005. Write down the dimension of the space of polynomials of degree at most 25.

[Marks: 1]

LA006. Write $\boldsymbol{c}$ in terms of standard basis vectors.

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 0 \\ 6 \end{bmatrix}$$

[MARKS: 1]

LA007. Place parentheses in the correct positions for the vector product equation to be true.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \times \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -24 \\ -6 \\ 12 \end{bmatrix}$$

[MARKS: 2]

LA008. Find the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ in degrees. ⟦*Hint*: Remember your trig values!⟧

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \boldsymbol{w} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 3 \end{bmatrix}$$

[MARKS: 2]

LA009.

$$\begin{bmatrix} e^{\pi i} & 0 & 7 \\ 0 & e^{2\pi i} & 0 \\ 6 & 0 & e^{4\pi i} \end{bmatrix}^2 =$$

To get the marks for this question the resulting matrix should have integer entries.

[MARKS: 2]

LA010. Consider the basis $\{2, x+3, (x-4)(x+5)\}$ for $\mathbb{R}[x]_2$. Determine the coordinates of $1 + x + 2x^2$ relative to this basis.

[MARKS: 3]

LA011. Does the set below form a basis for $\mathbb{R}^3$? .

$$\left\{ \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 18 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ 3 \end{bmatrix} \right\}$$

[MARKS: 3]

LA012. Determine the inverse (if it exists) of the matrix,

$$\boldsymbol{E} = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 2 & 0 \\ 6 & 0 & 4 \end{bmatrix}.$$

[Marks: 3]

LA013. Consider the following matrices in $M_{3,2}(\mathbb{R})$

$$\boldsymbol{C}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 6 \end{bmatrix}, \boldsymbol{C}_2 = \begin{bmatrix} 0 & 2 \\ 6 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{C}_3 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 6 \end{bmatrix}.$$

Are these matrices linearly independent?

[Marks: 4]

LA014. Hold on Vecna ... you seem to have turned my vector space upside down

Let $V = \{(x, y); x, y \in \mathbb{R}\}$. For $\boldsymbol{x}, \boldsymbol{a} \in V$, define the operation of "addition" to be

$$(x, y) \oplus (a, b) = -(x + a, y + b).$$

For all $\boldsymbol{x} \in V, \alpha \neq 0 \in \mathbb{R}$, define the "scalar multiplication" to be,

$$\alpha \odot \boldsymbol{x} = \frac{1}{\alpha}\boldsymbol{x}.$$

Is $(V, \oplus, \odot)$ a $\mathbb{R}^2$ vector space? If not, which axiom(s) fail(s) to hold?

[Marks: 4]

LA015. A typical day in MCS Level 3

Cassia is reaching hour nine in the MCS building and turns to her favourite pastime, distracting Freya.

Freya on the other hand, is hard at work. She wants Cassia to be quiet so sets her a task that will keep her occupied:

"The set of invertible $n \times n$ matrices with integer elements forms a group $GL_n(\mathbb{Z})$. If I give you an invertible matrix $\boldsymbol{A} \in M_n(\mathbb{Z})$, what condition is needed on $\boldsymbol{A}$ for it to lie in $GL_n(\mathbb{Z})$?"

Help Cassia find the condition(s) defining the group so she can continue distracting Freya.

[Marks: 4]

LA016. Place parentheses in the correct positions for the vector product equation to be true.

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 50604 \\ 220682 \\ 141080 \end{bmatrix} \times \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} \times \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

[Marks: 4]

LA017. Let $V = \mathbb{R}$. For $x, y \in V$, define the operation of "addition" to be

$$x \boxplus y = max(x, y).$$

For all $x, \alpha \in V$, define the "scalar multiplication" to be,

$$\alpha \boxdot x = x^{\alpha}.$$

Is $(V, \boxplus, \boxdot)$ a $\mathbb{R}$ vector space? If not, which axiom(s) fail(s) to hold?

[Marks: 4]

LA018. Let $V = \mathbb{C}^{\times}$. For $x, y \in V$, define the operation of "addition" to be

$$x \oplus y = \frac{1}{2} x \cdot y.$$

For all $x \in V, \alpha \in \mathbb{C}$, define the "scalar multiplication" to be,

$$\alpha \odot x = 0.$$

Is $(V, \oplus, \odot)$ a $\mathbb{C}$ vector space? If not, which axiom(s) fail(s) to hold?

[Marks: 4]

LA019. Let $V = \{(x, y, z); x, y, z \in \mathbb{R}\}$. For $\boldsymbol{x}, \boldsymbol{a} \in V$, define the operation of "addition" to be

$$(x, y, z) \boxplus (a, b, c) = (x + a, y + b, z + c + \frac{1}{2}(xb - ay)).$$

Define the operation of "scalar multiplication" as normal. Is $(V, \boxplus, \cdot)$ a $\mathbb{R}^3$ vector space? If not, which axiom(s) fail(s) to hold?

[Marks: 4]

LA020. Consider the matrix

$$\boldsymbol{H} = \begin{bmatrix} 1 & 6 \\ 7 & 2 \end{bmatrix}$$

Find the eigenvalues (with multiplicities) and eigenvectors of $\boldsymbol{G}$.

[Marks: 5]

LA021. You should know the drill by now...

Place parentheses in the correct positions for the vector product equation to be true.

$$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \times \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -720 \\ 360 \\ 0 \end{bmatrix}$$

[Marks: 5]

LA022. Don't worry it's not a typo

Place parentheses in the correct positions for the vector product equation to be true.

$$\begin{bmatrix}1\\1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\9\end{bmatrix}\begin{bmatrix}5\\0\\2\end{bmatrix}\times\begin{bmatrix}3\\6\\5\end{bmatrix}\times\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}-2340\\1320\\-100\end{bmatrix}$$

[Marks: 5]

LA023. Define the inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^2$ by

$$\langle \boldsymbol{z},\boldsymbol{w}\rangle = z_1\overline{w_1} + 5z_2\overline{w_2} + \frac{1+\mathrm{i}}{\sqrt{5}}z_1\overline{w_2} + \frac{1-\mathrm{i}}{\sqrt{5}}z_2\overline{w_1}$$

using this inner product, calculate the norm of the vector

$$\boldsymbol{u} = \begin{bmatrix}-14-\mathrm{i}\\ \sqrt{5}\mathrm{i}\end{bmatrix}.$$

[Marks: 5]

LA024. Decide which (if any) of the following bilinear functions defines an inner product:

1. $2x_1y_1 + x_2y_2 + 2x_3y_2 + x_2y_3$ on $\mathbb{R}^3$;
2. $10x_1y_1 + x_2y_2 - 15x_1y_3 - 15x_3y_1 - x_2y_3 - x_3y_2 + x_3y_3$ on $\mathbb{R}^3$;
3. $x_1y_1 - x_1y_3 - x_3y_1 + 2x_3y_3 + 4x_2y_2 + x_4y_4 + x_2y_4 + x_4y_2$ on $\mathbb{R}^4$.

[Marks: 5]

LA025. Cassia got bored of numbers

Same rules apply here place parentheses in the correct positions for the vector product equation to be true. To solve treat the letters like you would any variable.

$$\begin{bmatrix}J\\U\\N\end{bmatrix}\times\begin{bmatrix}A\\U\\G\end{bmatrix}\times\begin{bmatrix}O\\C\\T\end{bmatrix}=\begin{bmatrix}ACU+ANT-GNO-OU^2\\ACJ-CGN+JOU+NTU\\-AJT+CGU-GJO-TU^2\end{bmatrix}$$

[Marks: 5]

LA026. Determine the characteristic polynomial of the matrix

$$\boldsymbol{D} = \begin{bmatrix}1&5&3\\7&7&9\\6&1&7\end{bmatrix}.$$

[Marks: 6]

LA027. Determine determinant of the matrix,

$$\boldsymbol{F} = \begin{bmatrix} cos^2(x) & 7 & 3 & 1 & 0 & 1 \\ 0 & 3 & 9 & 4 & 2 & 1 \\ 0 & 0 & 1 & 4 & 5 & 2 \\ cos^2(x) & 10 & 12 & 2 & 7 & 2 \\ 0 & 3 & 10 & 8 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & sin^2(x) \end{bmatrix}.$$

[Marks: 6]

LA028. Determine determinant of the matrix,

$$\boldsymbol{G} = \begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 5 & 7 & 0 \\ 6 & 4 & 4 & 2 \\ 7 & 1 & 3 & 0 \end{bmatrix}.$$

[Marks: 6]

LA029. Determine a,b,c in the matrix

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix},$$

Such that the characteristic polynomial is $-\lambda^3 + 21\lambda^2 + 31\lambda + 22$.

[Marks: 8]

LA030. Consider the linear map $T : \mathbb{R}[x]_3 \to \mathbb{R}$ given by the mapping

$$T(p(x)) = p(-1) + \int_0^1 p(x)\,dx$$

Determine the nullity of the map T.

[Marks: 9]

LA031. Diagonalise the following matrix

$$\boldsymbol{I} = \begin{bmatrix} 6 & 2 & 3 \\ -13 & -5 & -11 \\ 4 & 2 & 5 \end{bmatrix}.$$

[Marks: 10]

LA032. Consider the Chebyshev-II operator

$$\mathscr{L}_{II} = (1 - x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}}{\mathrm{d}x}$$

on the space $C[1, -1]$ with respect to the inner product $(f, g) = \int_{-1}^{1} f(x)g(x)(1-x^2)^{1/2}\,\mathrm{d}x$. The Chebyshev-II polynomial of degree 4 is of the form $x^4 + ax^3 + bx^2 + cx + d$, find a,b,c and d.

[Marks: 10]

LA033. Consider the Hermite operator

$$\mathscr{L}_H = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x}$$

on the space $\mathbb{R}[x]$ with respect to the inner product $(f, g) = \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{e}^{-x^2}\,\mathrm{d}x$. The Hermite polynomial of degree 5 is of the form $x^5 + ax^4 + bx^3 + cx^2 + dx + e$, find a,b,c,d and e.

[Marks: 10]

LA034. For a function $T : \mathbb{R}[x]_3 \to \mathbb{R}[x]_3$ given by

$$T(p(x)) = p''(x) - xp'(x) + 2p(x)$$

that defines a linear map, write down the matrix T with respect to the standard basis of $\mathbb{R}[x]_3$ and explicitly give im(T) and ker(T).

[Marks: 11]

LA035. Solve the following system of first-order differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -3u - 2v - 6w,$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -8u - 3v - 12w,$$
$$\frac{\mathrm{d}w}{\mathrm{d}t} = 5u + 2v + 8w,$$

subject to initial conditions

$$u(0) = 1, v(0) = \frac{3}{2}, w(0) = -1.$$

You should hopefully get answers of the form $u(t) = ae^{bt} + ae^{ct}, v(t) = ae^{bt} + be^{ct}, w(t) = de^{bt} + de^{ct}$. Find a, b, c and d.

[Marks: 14]

LA036. Let V be the space $C[-2,2]$ equipped with the inner product

$$(f,g) = \int_{-2}^{2} f(t)g(t)\,\mathrm{d}t.$$

Let S be subspace of V spanned by $\{1, t, t^2\}$. Construct an orthonormal basis $\{g_1, g_2, g_3\}$ for S.

⟦*Hint*: 

⟧

[Marks: 15]

LA037. Let V be the space $C[-1,1]$ equipped with the inner product

$$(f,g) = \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t.$$

Let S be the subspace of V spanned by $\{1, t, t^2\}$. Construct an orthonormal basis $\{g_1, g_2, g_3\}$ for S, and find the function $h \in S$ closest to t^3.

[Marks: 17]

LA038. Who's to blame... Gram or Schmidt for the method, or Cassia for setting the question?

Use the Gram–Schmidt procedure to find an orthonormal basis for the subspace of $\mathbb{R}^4$,with the standard inner product, spanned by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \\ 2 \end{bmatrix} \right\}.$$

The resulting basis consists of vectors of the form

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{a}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{u}_2 = \frac{1}{\sqrt{a}\sqrt{b}} \begin{bmatrix} 1 \\ \sqrt{2}\sqrt{a} \\ -1 \\ 2\sqrt{2}\sqrt{a} \end{bmatrix}, \ \boldsymbol{u}_3 = \frac{1}{2\sqrt{b}} \begin{bmatrix} -c \\ 5 \\ c \\ -1 \end{bmatrix}$$

where a, b, c should be determined.

[Marks: 18]

LA039. For an integer $n > 1$. Let M be the set of all $n \times n$ matrices with entries only 0 and 1. What is the average determinant of a matrix in M?

[MARKS: 19]

LA040. A POWER TRIP TO NOWHERE

Let $\boldsymbol{A}$ be a $n \times n$ matrix such that $\boldsymbol{A}^k = 0$ for some $k > 0 \in \mathbb{Z}$, this is called a nilpotent matrix. Express the inverse of $\boldsymbol{A} + \boldsymbol{I}$ as a polynomial in $\boldsymbol{A}$. ⟦*Hint*: Your solution will be a summation⟧

[MARKS: 20]

Cassia Pearce
Cassia is a masters student at Durham University working on the finite Weil representation. She spends too much time procrastinating from her degree by rowing and being the current host of the Chalkboard Ultra podcast

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Probability I & Statistics I

Welcome to the **Probability and Statistics** section. Answer as many questions as you wish and simplify your answer where possible.

Note: You are not expected to do any "stupid" calculations. For example, if you have $\frac{17^7}{21}$ as an answer, you do not need to simplify any further.

~ *Louie Leventhall*

PS001. Consider a circle of diameter 1 fit just inside a square of length 1. Suppose you throw a dart at the square such that it is equally likely to fall on any point of the square. What is the probability the dart falls inside π.

[Marks: 1]

PS002. Below are a sample of 8 monthly rental prices for apartments in New York City.

3000 2000 3000 5000 500 5000 3000 1500

Find the median and interquartile range of this sample.

[Marks: 3]

PS003. **On your Marx, get set, Zo!**

Following on from the previous question, suppose a year after the sample was taken, the monthly rental prices have changed.

4000 3000 4000 6000 2500 6000 4000 2500

Newly inaugurated Mayor of New York City, Zohran Mamdani is investigating these 8 apartment prices as part of his *Freeze the Rent* campaign. To assist him, find the average rent increase from last year's prices to this year's.

[Marks: 3]

PS004. There are 100 passengers lined up to board an air-plane with 100 seats with each seat assigned to one of the passengers. The first passenger in line decides to sit in a randomly chosen seat (with all seats equally likely). Each subsequent passenger takes their assigned seat if available, and otherwise sits in a random available seat. What is the probability that the last passenger in line gets to sit in their assigned seat?

[Marks: 5]

PS005. Suppose a random variable X has mean 2 and variance 3. Find all the values of a and b such that $aX + b$ has mean 1 and variance 12.

[Marks: 6]

PS006. A typical Parliamentary Select Committee is made up of 11 members. Suppose 9 Labour MPs, 4 Conservative MPs and 3 Liberal Democrat MPs are sitting in a room. 11 of these MPs will be selected to join the committee*. Find the probability that exactly 5 members of the committee will be Labour MPs. Simplify as much as possible.

[Marks: 7]

PS007. Suppose that 20% of the students who took a certain test were from school A and that the arithmetic average of their scores on the test was 80. Suppose also that 30% of the students were from school B and that the arithmetic average of their scores was 76. Suppose, finally, that the other 50% of the students were from school C and that the arithmetic average of their scores was 84. If a student is selected at random from the entire group that took the test, what is the expected value of her score?

[Marks: 8]

PS008. A random sample of the members of a club is to be asked for their approval on a certain club matter. The secretary asks n club members of whom a proportion of 0.8 gives their approval on the matter. A 95% confidence interval for the proportion of members approving the club matter is found not to contain 0.9. Determine the least value of n. ⟦*Hint*: Use the Normal Tables in the Appendix⟧.

[Marks: 9]

PS009. Louie likes Baseball, so must you too

The Baseball Teams, the Milwaukee Brewers and the Chicago Cubs, are tied for first place with 7 games left in the season. Their last 7 games are against each other, so whoever wins the majority of games, wins the division. Suppose the Milwaukee Brewers have a 50-50 chance of winning each game, independent of all the other games. What is the probability the Milwaukee Brewers win the first game given they win the division?

[Marks: 9]

PS010. Losing my marbles

A bag contains a marble that, with equal probability, is either green or blue. A green marble is put in the bag and then a marble is taken out at random. The marble taken out is green. What is the probability that the remaining marble is green?

[Marks: 10]

PS011. Suppose X is a random variable that is uniformly distributed over the interval [-1,1]. Find $Var(X^3)$

[Marks: 10]

PS012. Consider a bivariate density function:

$$f(x,y) = 4xy, \quad 0 \le x, y < 1$$

Find $E(X|Y)$.

[Marks: 10]

*Usually Select Committee seats are allocated proportionally to each party's total number of MPs. But it is not always split evenly across every committee. See Select Committee Information

PS013. An Exponential Distribution has probability density function:

$$f(x|\lambda) = \lambda e^{-\lambda x} \text{ for } \lambda > 0,\ x \in [0, \infty).$$

Find the Moment Generating Function.

[Marks: 10]

PS014. Suppose X is a discrete random variable with a probability density function:

$$P(X = x) = \begin{cases} kx^2 & \text{for } 3, 4, 5 \\ 0 & \text{else} \end{cases}$$

for some constant k. Find k and hence $E(X)$.

[Marks: 10]

PS015. Suppose that 80% of all statisticians are shy, whereas only 15% of all physicists are shy. Suppose at a large gathering, 90% of the people are physicists and the other 10% are statisticians. If you meet a shy person at random at the gathering, what is the probability that the person is a statistician?

[Marks: 10]

PS016. Consider a bivariate density function:

$$f(x, y) = \frac{12}{7}(x^2 + xy), \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

Find $P(X > Y)$

[Marks: 10]

PS017. Let $X \sim N(\mu, \sigma^2)$, where μ is the mean and σ^2 is the variance. What is the probability that X is within 1 standard deviation from the mean? ⟦*Hint*: Use the Normal Tables in the Appendix⟧.

[Marks: 10]

PS018. Let X and Y be independent uniform random variables on $[0, 1]$. Suppose $Z = X + Y$. Find the correlation coefficient of Y and Z, denoted ρ_{YZ}.

[Marks: 11]

PS019. Suppose $x_1, x_2, \ldots, x_n$ is an i.i.d sample from a distribution with the pdf

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{if } x \ge 0 \\ 0 & \text{else} \end{cases}$$

where α is an unknown parameter. Find the MLE of α.

[Marks: 12]

PS020. A fair coin is flipped 30 times, with each flip being independent. Find the covariance between the numbers of heads among the first 20 flips and the last 20 flips.

[Marks: 15]

PS021. Suppose a random variable X has a probability density function:

$$f(x) = \begin{cases} \frac{3}{(1+x)^4} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$

Find $P(X > 1|X \leq 3)$.

[Marks: 15]

PS022. Suppose that X is a random variable that is distributed with a Poisson distribution such that $X \sim \text{Pois}(\lambda)$. Find $E(X!)$, assuming it is finite.

[Marks: 15]

PS023. The Jeffrey's Prior for θ is given as

$$J(\theta) \propto \sqrt{I(\theta)}$$

Where $I(\theta)$ is $-E(\mathscr{L}''(\theta))$ for $\mathscr{L}$ being the log-likelihood of $f(x|\theta)$. Suppose $f(x|\theta)$ defines a Bernoulli distribution such that $f(x|\theta) = \theta^x(1-\theta)^{1-x}$ for $x \in \{0, 1\}$. Find the Jeffrey's Prior.

[Marks: 15]

PS024. Consider a coin that when tossed shows heads with probability p. Determine the probability that the first heads will appear on the even-numbered tosses.

[Marks: 15]

PS025. A chicken lays a $\text{Pois}(\lambda)$ number N of eggs. Each egg independently hatches a chick with probability p. Let X be the number of chicks that hatch. Using the Law of Total Expectation, find $E(X)$, $E(NX)$ and hence $Cov(X, N)$.

[Marks: 16]

PS026. Suppose that $X_1, \dots, X_n$ are random variables such that the variance of each variable is 1 and the correlation between each pair of different variables is 1/4. Determine the value of $Var(X_1, \dots, X_n)$ in terms of n.

[Marks: 20]

PS027. Consider a bivariate density function:

$$f(x, y) = \frac{x+y}{3}, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 1$$

Find the covariance $Cov(X, Y)$.

[Marks: 20]

PS028. A bowl contains w white balls and b black balls. One ball is selected at random from the bowl, its colour is noted, and it is returned to the bowl along with n additional balls of the same colour. Another single ball is randomly selected from the bowl (now containing $w + b + n$ balls) and it is observed that the ball is black. Find the probability that the first ball selected was white.

[Marks: 20]

PS029. A Gamma Distribution with parameters $\alpha > 0$, $\beta > 0$ and $x \in [0, \infty)$ has a probability density function:

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \text{ where } \Gamma(\alpha) = (\alpha - 1)! \text{ for } a \in \mathbb{Z}.$$

Using LOTUS or otherwise, derive the exact form of the $E(X)$ and $Var(X)$.

[Marks: 20]

PS030. Scientists are attempting to do an experiment. They know that the outcome X is a random variable with a Bernoulli distribution such that $X \sim Ber(1/3)$. However, due to inaccuracies in their equipment, there is a Gaussian noise parameter $G \sim N(0, 1)$. So what is actually being measured is $M = X + G$. Assuming that X and G are independent, what is the probability that $X = 1$ if the measurement M is at most 4. ⟦*Hint*: Use the Normal Tables in the Appendix⟧

[Marks: 25]

Louie Leventhall
Louie is a recent graduate of Durham University who studied Mathematics and Statistics. He currently works as an Underwriter. He likes reading, doing cryptic crosswords and watching baseball.

in Louie Leventhall

Physics

Welcome to the final boss of first-year studies: the **Physics** section. Answer as many questions as you can, and for the love of dimensional consistency, do not forget your units — or you may share the fate of Throckmorton (see below).

Some of you may have noticed that physics ≠ mathematics, a claim which can indeed be proved (left as an exercise for the reader). While physics questions may contain more words than the average maths problem, please do not panic. Beneath the intimidating prose, physics is simply applied calculus and linear algebra, held together by assumptions, approximations, and a healthy disregard for exact answers. Good luck!

~ *Harry Graham*

PH001. Newton's $N^{\underline{th}}$ law

A skydiver jumps out of a plane and begins to accelerate. After some time, the skydiver reaches terminal velocity. Explain how this happens with reference to Newton's laws.

[Marks: 1]

PH002. The beginning of a longitudinal road

A wave propagates with a speed of 100 m/s with a wavelength of 5 m. Calculate the frequency of the wave. Is this wave electromagnetic?

[Marks: 1]

PH003. Gravitons in action?

An apple falls down 20 m from a tree, starting at rest. Calculate the speed of the apple just before it hits the ground. If we assume no air resistance, what would the speed of a feather be before hitting the ground in the same scenario?

[Marks: 1]

PH004. Newton's revenge

After the apple hit Newton on the head, Newton wasn't a happy man. In a fit of rage, big Isaac picked up the apple and chucked it at some angle α. Calculate the angle at which the apple can travel the furthest. You may find these formulas helpful.

$$v^2 = u^2 + 2as$$

,

$$s = ut + \frac{1}{2}at^2$$

[Marks: 3]

PH005. Throckmorton's Escape 2: Electric Boogaloo

Your annoying cousin Throckmorton, of mass 70 kg, has fallen foul of the law and is on the run....again. This time he has stolen a supercar, of mass 2000 kg, and is travelling at a speed of 180 kph. Calculate the total kinetic energy of Throckmorton and his newly acquired vehicle.

[Marks: 1]

PH006. The great escape

Unfortunately, Throckmorton was caught by the police and sent to a prison high in the mountains. He plans to escape justice by bungee jumping to freedom. Given a bungee cord of unstretched length 80 m, and assuming that air resistance and the mass of the bungee are negligible, calculate Throckmorton's velocity at the instant the bungee becomes taut. Assume the gravitational field strength $g = 10Nkg^{-1}$

[Marks: 1]

PH007. Hooke's Law And Order

As the bungee starts to stretch, Throckmorton begins to slow down until he travels a further 20 m until coming to a stop. Calculate the spring constant of the bungee, and hence the force from the bungee acting on Throckmorton. Assume air resistance and bungee mass are negligible.

[Marks: 2]

PH008. Second order consequences

At the instant Throckmorton comes to rest, the bungee snaps and he becomes one with gravity as he enters free fall. This time, however, the physics gods decide that air resistance may not be ignored, giving poor Throckmorton some hope of slowing down. Assuming the drag force is linear ($F_{drag} = -kv$ where v is velocity) and taking $t = 0$ at the moment when $v = 0$, find an expression for the displacement, $x(t)$, of Throckmorton's final descent as a function of time, t, and constant drag coefficient, k.

[Marks: 7]

Special relativity: index notation and Einstein summation convention

In physics, we are lazy, so instead of writing full column vectors and matrices out in full, we often write only their components. This is done using index notation. For example, in special relativity, we describe spacetime events using four-vector notation.

A general four-vector is written as

$$x^\mu = (x^0, x^1, x^2, x^3),$$

where the indices are not powers; they simply label the components of the vector (or, in the case of a matrix, its elements). For example, the four-position vector is given by

$$x^\mu = (ct, x, y, z),$$

where the first component is the time-like component (multiplied by the speed of light, c, to make it dimensionally consistent). The remaining components make up the usual three-position vector. If two indices in the same multiplication are repeated, then we must sum over all of the indices (from 0 to 3), for example:

$$A^\mu B^\mu = A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

To lower an index, one must multiply the four-vector by the Minkowski metric tensor, written as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

To lower an index, we contract the vector with the metric tensor:

$$x_\nu = g_{\mu\nu} x^\mu.$$

The metric tensor allows us to raise and lower indices. One index of the metric must be up and the other down, and the indices being summed over must appear once up and once down in the contraction. Computing the right-hand side of the above equation gives the new four-vector with a lower index of the form

$$x_\nu = (ct, -x, -y, -z),$$

and you will see why this is important in the next question.....
Lastly, in special relativity, you will need to be able to transform to different frames. You can do this by using Lorentz transformations, and they are given by

$$\begin{aligned} x \to x' &= \gamma(x - vt), & t \to t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ y \to y' &= y, & z \to z' &= z, \end{aligned} \qquad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

PH009. For the spacetimes, they are a changing

Bob is at rest with a position four-vector $x^\mu = (ct, 0, 0, 0)$. Dylan is in a moving frame relative to Bob with a velocity, v, and has four-position, $x^{\mu\prime}$. Show that the Minkowski scalar product of Bob's four-position with itself is frame invariant (ie, calculate $x^\mu x_\mu$ and compare it to $x^{\mu\prime} x'_\mu$ in Dylan's frame.

[Marks: 8]

PH010. No approximations in this integral (for once)

Consider the potential

$$V(r) = V_0 e^{-Mr}.$$

Calculate the matrix element of this potential for momentum eigenstates by performing the integral

$$\langle p' \,|\, V \,|\, p \rangle = \int_0^\infty V(r) e^{iqx} d^3x.$$

⟦*Hint*: You may use $\int x^k e^{-ax}\, dx = k!/a^{k+1}$.⟧

[Marks: 5]

PH011. Sir, this is not Newtonian anymore

Calculate the speed of a particle if its relativistic kinetic energy

$$E_k = (\gamma - 1)mc^2,$$

is equal to four times its rest mass. Give your answer as a surd and in terms of the speed of light, c.

[Marks: 4]

PH012. Just plug in and hope

Using the relativistic energy-momentum relation

$$E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4,$$

plug in the correspondence principle given as

$$\boldsymbol{p} \to -i\hbar\nabla, \qquad E \to i\hbar\frac{\partial}{\partial t},$$

to derive the free Klein-Gordon equation.

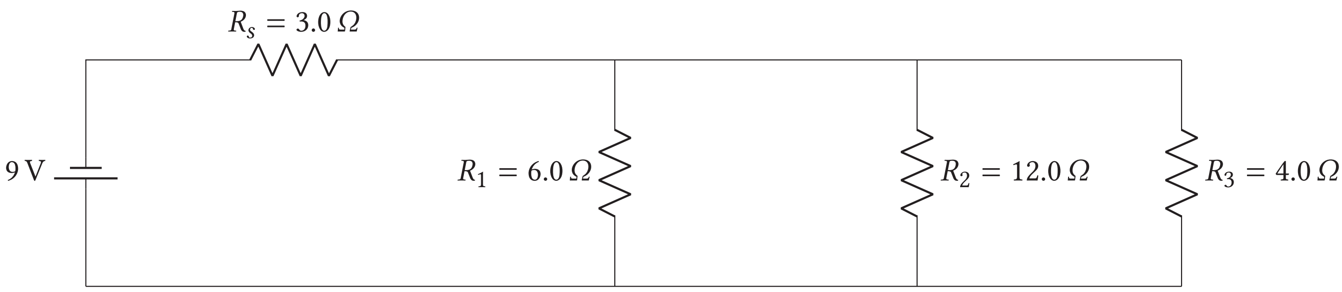
[Marks: 4]

Scalar Particles

Although we are not proving it here, the Klein-Gordon equation is the relevant equation of motion that describes scalar (or spin-0) particles. That doesn't mean the particle doesn't spin; it just means the particle is left invariant under any rotations. Examples of scalar particles include pions, kaons, axions (albeit hypothetical, though a strong candidate for dark matter) and the recently discovered Higgs boson.

PH013. We all have our potential differences

Consider a cell of potential difference $9V$ connected to a series resistor R_s, followed by three resistors R_1, R_2, R_3 in parallel.



Calculate the current that flows through R_s and calculate the potential difference that is supplied to each resistor in the circuit.

[Marks: 7]

PH014. The r^{-12} restraining order

The cohesive energy of a face-centred-cubic structured crystal is described by the Leonard-Jones potential

$$U(r) = -2\varepsilon\left[A_6\left(\frac{\sigma}{r}\right)^6 - A_{12}\left(\frac{\sigma}{r}\right)^{12}\right],$$

with $A_6 = 1$, $A_{12} = 32$. Use this expression to determine the value of the σ parameter in this case, in which the equilibrium atomic separation, r = 0.4 nm.

[Marks: 6]

PH015. Questions getting a bit harder? I'm sure you can ∂eal with it

A wave can be mathematically modelled by

$$\psi = A\cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t),$$

where A is the amplitude, $\boldsymbol{k}$ is the wave vector $\left(|\boldsymbol{k}| = 2\pi/\lambda\right)$ with λ the wavelength, t is time, and ω is the angular frequency $\left(\omega = 2\pi f\right)$.

Find the second partial derivative of ψ with respect to 3D space $\boldsymbol{x}$, and the second partial derivative with respect to t. Hence, show how these two derivatives are related.

[Marks: 6]

PH016. $\nabla\times\nabla\times\nabla\times\nabla\times\nabla\times\nabla\times\ldots$ please stop

Maxwell's equations describe the laws of electromagnetic phenomena. They are given as

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \qquad \text{(MAX1)}$$

$$\nabla\cdot\boldsymbol{B} = 0, \qquad \text{(MAX2)}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \qquad \text{(MAX3)}$$

$$\nabla\times\boldsymbol{B} = \mu_0 J + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \qquad \text{(MAX4)}$$

By taking the curl of (MAX3) and (MAX4), and setting the current/charge to 0 (as light), you should arrive at two second-order partial differential equations in terms of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. Using these equations and the equation you derived in the previous problem, find the speed of light in terms of constants ε_0 and μ_0. What can we say about the nature of light from these differential equations? ⟦*Hint*: You may need the vector identity $\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$⟧

[Marks: 11]

PH017. The determinant one took him out!

The Standard Model gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$, where $SU(3)_c$ is the gauge symmetry of Quantum Chromodynamics (quark and gluon interactions; c denotes colour charge).

Elements of a continuous group may be written in the form

$$U(\alpha^a) = \mathrm{e}^{\mathrm{i}\alpha^a T^a},$$

where $U(\alpha)$ are 3×3 matrices (in $SU(3)$), α^a are continuous parameters and T^a are the generators of the group (also known as the Gell-Mann matrices).

Using the defining properties of $SU(3)$:

$$U^\dagger U = I, \qquad \det(U) = 1,$$

determine the total number of independent generators of $SU(3)$. Hence state how many different types of gluons are allowed in the Standard Model.

[Marks: 10]

PH018. Born of commutator trauma

Two Gell-Mann matrices λ^1 and λ^2, where $T^a = \lambda^a/2$, are given as

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The generators of $SU(3)$ satisfy the Lie algebra

$$[T^a, T^b] = T^a T^b - T^b T^a = \mathrm{i} f^{abc} T^c,$$

with $f^{123} = 1$. Using this information, determine λ^3. Hence find the eigenvalues and normalised eigenvectors of λ^3.

[Marks: 10]

PH019. Gauge bosons, electroweak edition

The remaining gauge groups in the Standard Model are $SU(2)_L$ and $U(1)_Y$. These together describe the electroweak interaction, which contains both the electromagnetic and weak interactions.

Match each gauge group to the interaction it primarily gives rise to before electroweak symmetry breaking, and justify your answer using the number of gauge bosons.

⟦*Hint*: The electromagnetic interaction has one hypercharge gauge boson B_μ, while the weak interaction has three gauge bosons W^1, W^2, W^3 before symmetry breaking.⟧

[Marks: 6]

PH020. Group therapy session

Abelian groups are determined if the elements of the groups commute, and non-Abelian if they don't. Determine if $SU(3)$ and $U(1)$ are Abelian or non-Abelian.

[Marks: 2]

PH021. Ways to Ruin a Perfectly Good Derivative

The free energy density of an ordered phase of matter is given by

$$f = (\nabla\phi)^\dagger(\nabla\phi) - \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2,$$

where ϕ is a complex field and μ and λ are positive real coefficients. Determine the effect on the free energy under a global $U(1)$ transformation ($\phi \to \phi \mathrm{e}^{\mathrm{i}\alpha}$) and under a local $U(1)$ transformation ($\phi \to \phi \mathrm{e}^{\mathrm{i}\alpha(x)}$). State the additional terms that arise if the free energy is not left invariant.

[Marks: 5]

PH022. You can get this done faster than the speed of light

The electromagnetic field strength tensor is defined as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Compute $F^{\mu\nu}F_{\mu\nu}$ and simplify as much as possible.

[Marks: 12]

PH023. The conservation of pain

The dual field strength tensor is defined as

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma},$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in four dimensions. Construct $\widetilde{F}^{\mu\nu}$ by calculating its elements and use this to calculate $\widetilde{F}^{\mu\nu}F_{\mu\nu}$

[Marks: 15]

PH024. Physicists Were Born Approximating

The first Born approximation provides a method for computing the scattering amplitude of a particle interacting with a weak potential. In scattering theory, the wavefunction far from the interaction region can be written in the asymptotic form

$$\psi(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + f(\theta,\phi)\,\frac{\mathrm{e}^{ikr}}{r},$$

where $f(\theta,\phi)$ is the scattering amplitude and $\mathbf{k}$ is the incident wavevector. The first term represents the incoming plane wave, while the second term describes the outgoing spherical wave produced by the scattering process.

For a spherically symmetric potential $V(r)$, the first Born approximation gives the scattering amplitude as

$$f_B(\theta, \phi) = -\frac{2m}{\hbar^2 \Delta} \int_0^\infty r\, V(r) \sin(\Delta r)\, \mathrm{d}r,$$

where

$$\Delta = 2p \sin\left(\frac{\theta}{2}\right)$$

is the momentum transfer and $p = \hbar k$ is the magnitude of the incident momentum. Consider the potential

$$V(r) = \frac{\hbar^2 U_0}{2m} \frac{4d^3}{\pi(r^2 + d^2)^2},$$

where U_0 and d are positive constants representing the height and width of the potential, respectively. Using the first Born approximation, calculate the scattering amplitude $f_B(\theta, \phi)$ for this potential.
⟦*Hint*: You may find the following integral useful

$$\int_{-\infty}^{\infty} \frac{\cos(\Delta x)}{x^2 + d^2}\, \mathrm{d}x = \frac{\pi \mathrm{e}^{-\Delta d}}{d}.$$

Or you might not. Whatever floats your boat.⟧

[Marks: 16]

PH025. Anything is linear if you are brave enough

For the Yukawa potential

$$V(r) = \frac{\hbar^2 U_0}{2m} \frac{\mathrm{e}^{-r/d}}{r},$$

the scattering amplitude in the first Born approximation is

$$f_B = -\frac{U_0 d^2}{1 + d^2 \Delta^2}.$$

Calculate the differential cross section for the Yukawa potential and expand it for small scattering angles θ, keeping only the leading order in θ.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f_B(\theta, \phi)|^2$$

[Marks: 3]

PH026. Localised suffering in four dimensions

The Feynman propagator for a scalar field is given by

$$G_F(x - y) = \mathrm{i} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i} p_\mu (x-y)^\mu}}{p^2 - m^2 + \mathrm{i}\varepsilon}.$$

Evaluate the integrals

$$\int \mathrm{e}^{-(k_\mu x^\mu)^2} \left(\frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x_\alpha} + m^2\right) G_F(x - y)\, \mathrm{d}^4 x,$$

and

$$\int e^{-(k_\mu x^\mu)^2}\left(\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y_\alpha}+m^2\right)G_F(x-y)d^4x.$$

[Marks: 17]

PH027. You're cooked, LOL

A scalar particle Φ decays into two scalar particles φ and φ', $\Phi(q) \to \varphi(p) + \varphi'(p')$ with $q' = (M, \mathbf{0})$. Calculate the decay rate of this process given by the integral below

$$\Gamma_\Phi = \frac{1}{2M}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_p}\frac{d^3\boldsymbol{p}'}{(2\pi)^3 2E_{p'}}(2\pi)^4\delta^3(\boldsymbol{q}-\boldsymbol{p}-\boldsymbol{p}')\delta(M-2m_\varphi)\kappa^2.$$

Where M is the mass of Φ, m is the mass of φ and φ', κ is a constant, q and p/p' are the four-momenta of Φ, φ and φ' respectively. You may find the relativistic energy-momentum relation helpful,

$$E^2 = \boldsymbol{p}^2 + m^2.$$

⟦*Hint*: Start by integrating either $\boldsymbol{p}$ or $\boldsymbol{p}'$ using the Dirac delta with the 3-momentum inside. Then you may need to change to spherical polar co-ordinates.⟧

[Marks: 20]

PH028. Boss Fight: Phase Two

Consider two fermion fields $t(x)$ and $b(x)$ with masses m_t and m_b and a complex scalar field ϕ^* with mass m. The couplings for these particles are given by the Lagrangian

$$\mathscr{L} = -y\bar{t}\phi - y\phi^*\bar{b}t.$$

Approximating $m, m_b \to 0$, calculate the invariant matrix element, $\langle|\mathscr{M}|^2\rangle = \sum_{s_b,s_t}|\mathscr{M}|^2$ for $t(p) \to b(k)\phi(k')$ and the decay rate in t's rest frame $p = (m_t, \mathbf{0})$ by calculating

$$\Gamma_t = \frac{1}{2m_t}\frac{1}{2}\sum_{s_b,s_t}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_p}\frac{d^3\boldsymbol{p}'}{(2\pi)^3 2E_{p'}}(2\pi)^4\delta^4(p-k-k')|\mathscr{M}|^2$$

⟦*Hint*: The following relation will be needed $\sum_s[u(p,s)]_a[\bar{u}(p,s)]_b = (\gamma_\mu p^\mu + m)_{ab}$⟧

[Marks: 25]

Harry Graham

Harry is a PhD student at Durham University in the Institute for Particle Physics Phenomenology (IPPP) under the supervision of Frank Krauss and Martin Bauer. His research focuses primarily on the Monte Carlo event generator SHERPA (Simulation of High-Energy Reactions of Particles), as well as modelling dark matter background fields. Outside of research, he enjoys playing badminton, relaxing, and designing exam questions of such difficulty that they briefly cause students to reconsider their life choices.

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6

Special

SP001. Sam didn't spend 10 hours on the appendix for nothing

As a team, come up with a plausible UK registration plate (using any of the systems) that spells out a 'maths word' distinct from the other teams'. For example,

CYC 70ID

M345 URE

TH30 REM

would be valid answers. Points will be awarded to the team with the most unique/fun/clever answer during the final hour of *Le Maths*.

[Sparks: 100]

SP002. Find all $x \in \mathbb{Z}$ such that $x^x \equiv 2 \pmod 5$.

[Sparks: 15]

SP003. Luci would not like to be associated with '67', it just made a good question

Find the missing values L, U, C, I such that $L \neq U \neq C \neq I \neq \{6, 7\}$.

$$\begin{array}{cccccc} & & & 6 & 7 & I \\ & & \times & 6 & L & 7 \\ \hline L & U & 6 & 7 & C & I \end{array}$$

[Sparks: 7]

SP004. Consider an infinite plane populated by an infinite number of parallel lines of infinite extent, each separated by a distance of one unit. Suppose we throw a tetrahedral die of side length one. What is the probability that the die will land on at least one line? What is the probability that a cube-shaped die of side length one will land on at least one line?

⟦*Hint*: You may wish to solve a simpler problem first! Given the same infinite plane of parallel lines, what is the probability that a randomly thrown stick of unit length will land on at least one line?⟧

[Sparks: 12]

SP005. Andreo has created n clones of himself and devised a Sisyphean way to torture them all simultaneously. He places each clone on a different vertex of a regular n-gon of side ℓ. At time $t = 0$, each clone starts chasing their counter-clockwise neighbour with the same constant speed v, always making sure that their velocity is pointing towards their chased neighbour. After how long do the clones meet each other? Is there a shape for which this time is exactly ℓ/v?

[Sparks: 20]

SP006. If Euclid and Euler could do it, then so can you!

Take as definition that a number is **perfect** if and only if its sum of divisors is twice its value. Prove the following statements:

Sufficiency. Given a Mersenne prime $2^p - 1$, the quantity $2^{p-1}(2^p - 1)$ is **even** and **perfect**.

Necessity. Given an even perfect number, it **must** be of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime.

⟦*Hint*: You may wish to use the fact that if the greatest common divisor of $a, b \in \mathbb{N}$ is 1 then

$$\text{sum of divisors of } (a \times b) = (\text{sum of divisors of } a) \times (\text{sum of divisors of } b).$$

Or not. It's your call.⟧

[Sparks: 20]

SP007. It all adds up to...

Evaluate

$$\sum_{k=0}^{67} \left(-\frac{6}{7}\right)^k \binom{67}{k} \left[\frac{7}{6}k + 469\right].$$

[Sparks: 25]

SP008. This really IS the final countdown

We can say for sure that about 91% of Countdown numbers games are possible. For example, the one below has no solution:

But how close can you get?

⟦*Hint*: You get for free that it's more than one or two away.⟧

[Sparks: 12]

SP009. Is that an error?

Find the first five terms in the Taylor expansion of the error function,

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt.$$

[Sparks: 7]

SP010. James Grime's *Dice Hard*

James Grime is the host of the up-and-coming carnival game *Dice Hard*. In it, you pay £2 and roll seven dice. You then arrange the seven dice to make a seven-digit number. The prizes are awarded by conjuring a number that is a multiple of the prize money (up to £7), as summarised in this table:

Multiple	Prize
2	£2
3	£3
4	£4
5	£5
6	£6
7	£7

The catch? *You must choose your prize* ***before*** *you roll.*

Wanting to play this game multiple times, what prize should you choose to maximise winnings?

[Sparks: 15]

SP011. How many positive integers less than 1000 have a digit sum equal to $n < 10$?

[Sparks: 5]

SP012. The points $2 + 3i$ and $1 - 4i$ are diagonally opposite vertices of a regular hexagon in the Argand diagram. Find the other vertices.

[Sparks: 6]

SP013. Find the exact value of $\cos(2\pi/5)$.

[Sparks: 7]

SP014. SURPRISE!!!

It's Lewis' birthday... but ssh! We're hosting a surprise maths-themed party and we've invited all N of his friends (I say we, it was all Lexie's idea). Now, I know what you're thinking: a maths-themed birthday party would be super boring. Luckily, Lexie came up with the following game to make things more interesting:

"When Lewis enters the room, let's shout our ages. I will lead him out of the room and he then has to decide whether or not there are two people whose ages differ by a multiple of the number of people left in the room."

Will it be possible for Lewis to come up with such a pair for any N?

[Sparks: 10]

SP015. I have discovered a truly marvellous proof of this, which this napkin is too messy to contain

After the horrific display of events at Lewis' surprise maths-themed party (as claimed by Throckmorton, who wasn't actually meant to be invited), Lexie wants to make it up to Lewis by making up a new game to be played on their maths-themed double date with Sam and Andreo.

They settle on YO! Sushi as their restaurant of choice. At the start of the meal, Sam and Andreo choose different pieces of sushi to represent 1, 10, 100, 1000 and so on. Combining different numbers of sushi will then represent any natural number N.

Lexie then points out that if you take the number of sushi pieces away from the number that the sushi represents, you will always get a multiple of nine. She writes down a proof of why this happens on her napkin, but Lewis (the messy eater that he is) uses this napkin to wipe soy sauce off his chin.

Help Lexie remember the proof by writing it down!

⟦*Note*: An extra 5 marks are on offer if the proof is handed in on a napkin.⟧

[Sparks: 10]

SP016. Find the constant $\lambda \in \mathbb{Q}$ such that $\displaystyle\int_0^1 \frac{\log(s)}{\sqrt{s}} e^{-r^2 s}\, ds = \lambda \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n}}{n!(2n+1)^2}$.

[Sparks: 12]

SP017. Supervised study

With his newfound research grant money, Peter Wyper sends a shockwave through the department by buying a brand-new motorcycle. He wants to show it off to Adam (for street cred), Sam (for coolness points) and to Luci (because we needed a girl in this question for EDI purposes) by offering them a lift to the Vic for a pint. The problem is, Peter's motorcycle only has room for two riders: Peter and one passenger, Adam or Sam or Luci. If Peter takes Adam with him, Sam will distract Luci from her research by pestering her to write more Calculus questions for *Le Maths*. It's a valid procedure; she hadn't started until five days before the event. If he takes Sam, Luci will distract Adam from his teaching via her greatest superpower: persistent yapping. Only when Peter is present are Adam and Luci safe from the drama that each of their academic enemies causes. All the same, Peter manages to find a way to escort Luci, Adam and Sam to the pub without any mention of troublesome behaviour. What is Peter's office number?

[Sparks: 5]

SP018. Find an explicit expression expression for $\boldsymbol{A}^{100}$ given that $\boldsymbol{A} = \begin{pmatrix} 2 & -3 \\ 2 & -5 \end{pmatrix}$.

[Sparks: 7]

SP019. Find a matrix $\boldsymbol{B}$ satisfying $\boldsymbol{B}^5 = \begin{pmatrix} -67 & 198 \\ -33 & 98 \end{pmatrix}$.

[Sparks: 7]

Uptown Funke

When invited to submit a bonus lecturer question, our beloved Overlord of Analysis and Number Theory Prof. Jens Funke provided a flurry of such suitable questions.

We whittled it down to one candidate (spoiler alert: get ready for 3am), but the rest still seemed like fun. So we included them here.

SP020. Diophantus ($3^{\underline{rd}}$ century CE), *Arithmetica*, Problem 24, Book IV

> *"To divide a given number into two numbers such that their product is a cube minus its side."*

That is, for a positive integer n, find a solution to

$$y(n-y) = x^3 - x$$

with $x, y > 0$ *positive rational* numbers. Take $n = 6$.

[Sparks: 15]

SP021. What is the final digit of 777^{333}?

[Sparks: 10]

SP022. Show that $12^x = x^{72}$ has three solutions, and find one of them explicitly.

[Sparks: 15]

SP023. For any $n \in \mathbb{N}$, find the sum

$$-1 + (2+3+4) - (5+6+7+8+9) + (10+\ldots+16) - (17+\ldots+25) + \ldots + (-1)^n \left([(n-1)^2+1] + \ldots + n^2\right).$$

That is, the sum from 1 to n^2 with the overall sign changing between each intermediate square.

[Sparks: 8]

SP024. Consider five positive integers such that the sum of any three of them is divisible by the sum of the remaining two. For example, 1, 1, 1, 1, 2 have this property.

Show that there exist no pairwise-different five numbers (so at least two numbers must be the same) with that property.

[Sparks: 10]

SP025. Using the same setup as previous, show that at least three of the numbers have to be the same.

[Sparks: 10]

SP026. Find all quintuples with this property.

[Sparks: 10]

SP027. The road to Abel's Limit Theorem

Let b_n be a zero sequence. Show $\lim_{x\to 1^-}(x-1)\sum_{n=1}^{\infty} b_n x^n = 0$.

[Sparks: 15]

SP028. Let a_k be another sequence such that its associated series $\sum_{k=1}^{\infty} a_k$ converges to some $s \in \mathbb{R}$. Set $s_n{=}\sum_{k=1}^{n} a_k$ and $b_n = s - s_n$.

Express $(x-1)\sum_{n=1}^{\infty} b_n x^n$ in terms of $\sum_{k=1}^{\infty} a_k x^k$ and show $\lim_{x\to 1^-}\sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} a_k = s$.

[Sparks: 15]

SP029. Give me a good reason as to why this doesn't all simply follow from the continuity of power series.

[Sparks: 5]

SP030. Given $x \in \mathbb{R}$, let $s(x)$ be the distance to the nearest integer; $s(x)$ is therefore continuous and 1-periodic given on $[0,1]$ by

$$s(x) = \begin{cases} x & 0 \leqslant x \leqslant 1/2, \\ 1-x & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Show that $f(x) := \sum_{k=0}^{\infty} \frac{s(4^k x)}{4^k}$ is continuous on $\mathbb{R}$, and compute $\int_0^1 f(x)\,\mathrm{d}x$.

[Sparks: 10]

SP031. Using f defined as above,

(J) Show that f is not differentiable at $x = 0$.

(F) Show that f is not differentiable at any $x \in \mathbb{R}$.

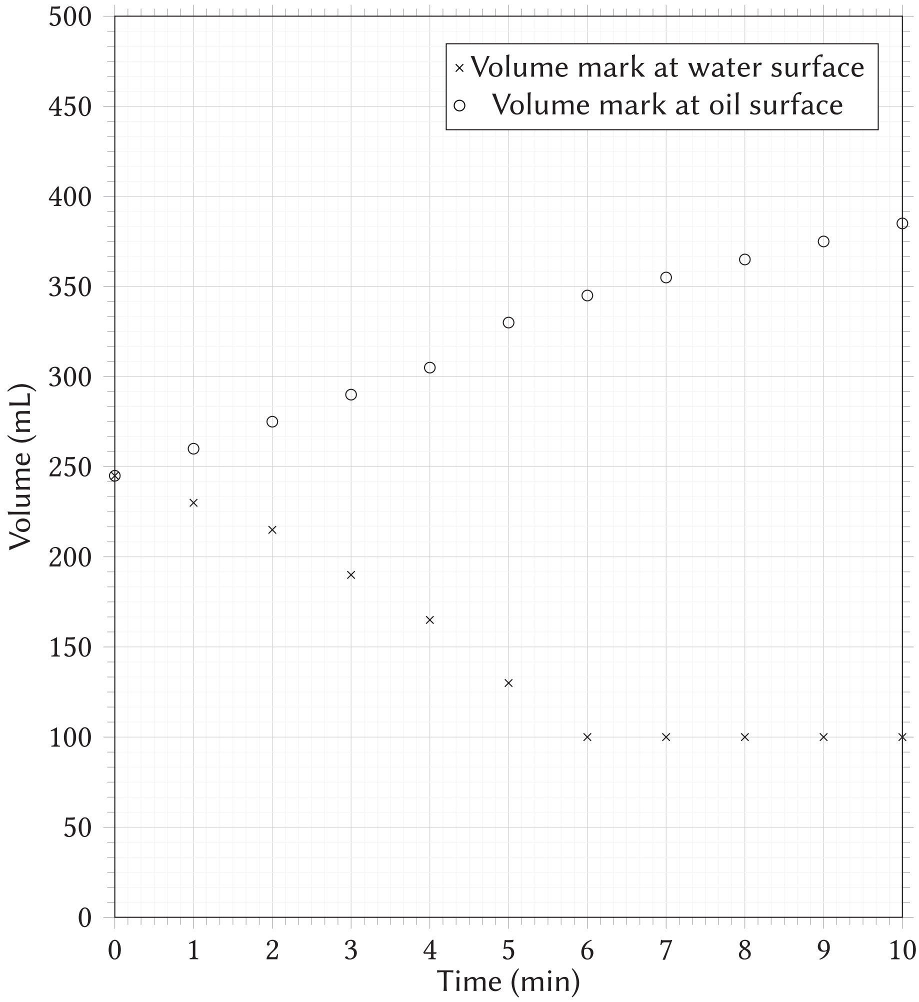
⟦*Hint*: Consider the sequence $x_n = 4^{-n}$.⟧

[Sparks: 20]

End of *Uptown Funke* section

SP032. Andreo had a rubber ducky and too much time on his hands

A graduated cylindrical beaker is partially filled with water; a rubber duck floats at the surface. Oil is poured into the graduated cylinder at a slow, constant rate, and the volume marks corresponding to the surface of the water and the surface of the oil are recorded as a function of time. The resulting plot is shown below. Water has a density of 1.00 g/ml; the density of air is negligible, as are surface effects. Find the density of the oil.



[Sparks: 20]

SP033. It's witching hour... time for some spooky maths!

Find:

(a) $\begin{pmatrix} 4 & 7 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 9 & 7 \\ 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 9 \\ 8 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 23 & 10 & 12 \\ 69 & 30 & 36 \\ 92 & 40 & 48 \end{pmatrix}^2$

(c) $\begin{pmatrix} 2 & 1 & 3 & 1 \\ 6 & 3 & 9 & 3 \\ 2 & 1 & 3 & 1 \\ 6 & 3 & 9 & 3 \end{pmatrix}^2$

(d) $\begin{pmatrix} 1000008121 & 1118627180 \\ 8045634086 & 8999991880 \end{pmatrix}^2$

⟦*Note*: If this question is submitted during witching hour (3am – 4am) you will earn a bonus 12 marks.⟧

[Sparks: 13]

SP034. The infamous SP034

Find:

$$L = \lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

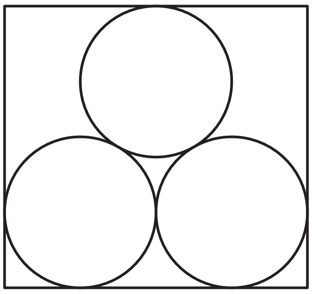
⟦*Fun fact*: This question appeared on the first *24 Hours of Le Maths* paper back in 2022!⟧

[Sparks: 100]

SP035. What is the total digit sum of the integers from 0 to 999 inclusive?

[Sparks: 5]

SP036. A rectangle is drawn around three equal circles as shown in the diagram below. How many times bigger in area is the rectangle compared to one of the circles?



[Sparks: 8]

SP037. When the delirium sets in … the forehead writing begins?

Cassia, Luci and Sam have been marking le maths questions for 20 hours, they've reached the point of tiredness that every statement said is true.

Luci decides to start writing whole numbers on Cassia and Sam's foreheads (because why wouldn't she?). The difference between the two numbers is one: either Cassia's number is one larger than Sam's, or Sam's is one larger than Cassia's. Each of Cassia and Sam can see the number on the other's forehead, but cannot see their own.

Luci writes a number on Cassia and Sam's foreheads, and says "Each of your numbers is at least 1. The difference between the numbers is 1."

Cassia then says "I know my number."

What must Cassia's number be?

[Sparks: 5]

SP038. Luci now writes new numbers on the foreheads, and says "Each of your numbers is between 1 and 10 inclusive. The difference between the numbers is 1. Cassia's number is a prime."

Cassia then says "I don't know my number."

Sam then says "I don't know my number."

What is Cassia's number?

[Sparks: 6]

SP039. Luci is starting to get bored of this game but decides to do it once more. "Each of your numbers is between 1 and 10 inclusive. The difference between the numbers is 1."

Cassia then says "I don't know my number, is it square?"

Luci says "If I told you that, you would know your number."

Sam says "I don't know my number."

What is Cassia's number?

[Sparks: 5]

SP040. Clare, trying to engage more of her SMB students in lectures, carries with her a bag that contains number cards with the digits 1–9 written on them. For each student she comes across, she asks them to take out three cards and write down all possible one, two and three digit numbers they can make using those cards. Each student then adds up each of the numbers they made. As it turns out, the final number that everyone obtains is divisible by p^2 for some p. Find p.

[Sparks: 8]

SP041. Solve the following for x:

$$(56x^3 - 3963x^2 + 14247x - 7370)^{(x^3-47x^2+719x-3553)} = 1.$$

⟦*Hint*: You may want to use the cubic formula that solves $ax^3 + bx^2 + cx + d = 0$:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt[2]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$+\sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt[2]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}.$$

Or not. It's your call.⟧

[Sparks: 13]

SP042. Write down the matrix $\boldsymbol{A} \in M_5(\mathbb{R})$ whose $(i,j)^{\underline{th}}$ entry is

$$A_{ij} = (-1)^{i+j} 2^i j \quad \text{for} \quad 1 \leqslant i, j \leqslant 5.$$

[Sparks: 4]

SP043. What is the volume of a 4D unit hypersphere, to seven decimal places?

[Sparks: 7]

SP044. What about the volume of an n-dimensional hypersphere of arbitrary radius R, in terms of π?

⟦*Hint*: $\Gamma(1/2) = \sqrt{\pi}$.⟧

[Sparks: 9]

SP045. U-G-L-Y, you ain't got no fire fly

You are a caveman mathematician, wanting to send a message to your friend and fellow cave-mathematician Ug in the next mudhole over. This being 10,000 BC, your only means of communication is smoke signals. You can burn two types of wood: one burns white, one black. To send a message with n colours, you need n fires. You need fires to roast mammoths, so you don't want to use more than necessary.

This begin 10,000 BC, you are also the first caveman mathematician who is investigating 'totally multiplicative' functions from the natural numbers which take values in the set $\{0, 1\}$. These are functions such that

$$f(pq) = f(p)f(q)$$

for any $p, q \in \mathbb{N}$, with $f(1) = 1$. You have such a function f in mind, and want to communicate the values $f(1)$, $f(2)$, ..., $f(n)$ to Ug. Ug does not know any of this, but does know how to translate from a series of smoke signals to Uglish, the language you both speak.

Show that there is a constant C such that, no matter what the value of n, you can send enough information to Ug for him to reconstruct the sequence $f(1)$, $f(2)$, ..., $f(n)$ using at most $C \cdot n/\log(n)$ fires.

⟦*Hint*: In what famous theorem does the expression $n/\log(n)$ show up?⟧

[Sparks: 14]

SP046. There aren't many nice solutions to the Navier–Stokes equations, but here's one

In cylindrical coordinates (r, θ, z), an axisymmetric fluid velocity field

$$\boldsymbol{u}(r, z, t) = u_r(r, z, t)\hat{\boldsymbol{e}}_r + u_z(r, z, t)\hat{\boldsymbol{e}}_z$$

is *incompressible* if it satisfies $\nabla \cdot \boldsymbol{u} = 0$. This condition can automatically be satisfied if we write $\boldsymbol{u}$ in terms of a *Stokes stream function* $\psi(r, z, t)$ as such:

$$\boldsymbol{u}(r, z, t) = \nabla \times \left(\frac{\psi}{r}\hat{\boldsymbol{e}}_\theta\right).$$

Find ψ for the so-called *Poiseuille* flow, $\boldsymbol{u} = (\alpha^2 - r^2)\hat{\boldsymbol{e}}_z$, where α is a constant.

⟦*Hint*: You may wish to use the following in cylindrical coordinates:

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\boldsymbol{e}}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{e}}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{\boldsymbol{e}}_z,$$

or derive it yourself.⟧

[Sparks: 15]

SP047. Find all values of $z \in \mathbb{C}$ such that $\cos(z) = 2$.

[Sparks: 6]

SP048. Find

$$\int \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\, \mathrm{d}a.$$

[Sparks: 10]

Lecturer Questions

BONUS TIME!

Here is the comprehensive list of the bonus lecturer questions that appear during the latter half of the competition.

LEC001. SLEEPING PIRATES

N pirates, where N is an integer greater than 1, land on a desert island that is a perfectly flat circular disc. The pirates are unhappy if they sleep near another pirate, because they fear that they may be murdered in their sleep. To calculate the total **Unhappiness** of all the pirates, sum the reciprocals of the $N(N-1)/2$ distances between all pairs of pirates.

If the pirates sleep at the edge of the island on the vertices of a regular N-gon, then (in units in which the island has radius one) the total **Unhappiness** for $2 \leq N \leq 20$ is given in the table across (to two decimal places). Find a value of N and an associated sleeping arrangement that is better than the N-gon at the edge of the island, in that it has an **Unhappiness** that is lower than the value given in the table.

The number of marks awarded for this question is a strictly decreasing function of your value of N.

Paul Sutcliffe

[MARKS: 100]

N	**Unhappiness**
2	0.5
3	1.73
4	3.83
5	6.88
6	10.96
7	16.13
8	22.44
9	29.92
10	38.62
11	48.58
12	59.81
13	72.35
14	86.22
15	101.45
16	118.06
17	136.07
18	155.50
19	176.37
20	198.69

LEC002. SAME HEAT, DIFFERENT SEATS

Blinky lives on the beautiful planet Circulia, which some people describe as a one-dimensional circle (though Blinky often bristles at the simplistic description of their beloved world!). They are a well-respected member of the Temperatorists Society, a society of like-minded people who enjoy the occasional temperature reading analysis across Circulia.

Unfortunately, Dr. Brumpy, Blinky's old childhood friend-turned-arch-nemesis (who is not a doctor at all!), is using his political influence to try and outlaw the Temperatorists Society—claiming that temperature reading in Circulia is meaningless and divisive as each place has their unique temperature identity.

Help Blinky and the Temperatorists Society refute Dr. Brumpy's claims by showing that not only there are always two places in Circulia with the exact same temperature, but there are infinitely many pairs of places that share the same temperature. From a mathematical point a view you can assume that Circulia is represented by the unit circle and that the temperature is a continuous function from this circle to the real line.

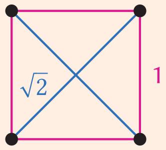
Amit Einav

[MARKS: 100]

LEC003. Distance makes the heart grow fonder, two distances doubly so

2-distances

A set of points S in $\mathbb{R}^n$ is called a *2-distance set* if the Euclidean distances between pairs of distinct points in S take at most two values. For instance, the vertices of a unit square form a 2-distance set in $\mathbb{R}^2$ with 4 points, since the only distances are the sides of length 1 and the diagonals of length $\sqrt{2}$:



a) Find a 2-distance set in $\mathbb{R}^2$ with 5 points.

b) Find a 2-distance set in $\mathbb{R}^3$ with 6 points.

c) Let $\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_n$ be the standard unit vectors in $\mathbb{R}^n$ and consider the set of points $\hat{\boldsymbol{e}}_i + \hat{\boldsymbol{e}}_j$ where $1 \leq i < j \leq n$. Show that this is a 2-distance set in $\mathbb{R}^n$ with $\binom{n}{2}$ points.

d) Show that there is a 2-distance set in $\mathbb{R}^n$ with $\binom{n+1}{2}$ points.

⟦*Hint*: look at the construction in part c). What linear algebra property do all the points have in common?⟧

Dan Evans

[Marks: 100]

LEC004. Prattle beyond the stars

An alien race visits Earth. This race, which looks strangely like rhino-headed people in shiny plastic packaging, speaks a language with exactly 21 syllables – "Bo", "Co", "Do", "Fo", etc, all the way up to "Zo". Every word in the alien language is made up of one or more of these 21 syllables.

a) As of January 2026, the Oxford English Dictionary contains 520,779 words. If we assume the alien language has the same number of words, what is the minimum length in syllables of the longest word in the alien language?

Every alien has a name which is precisely four syllables long. The alien's home world has two inhabited continents. Aliens from the Southwest continent all have names that contain exactly three distinct syllables, eg "Jo-jo-ro-bo" or "Xo-do-so-do". Every four-syllable word with precisely three distinct syllables is considered a name on the Southwest continent. Aliens from the Southeast continent all have names that contain at most two distinct syllables, eg "Mo-mo-go-mo" or "fo-mo-fo-mo" or "Yo-yo-yo-yo". Every four-syllable word with at most two distinct syllables is considered a name on the Southeast continent.

b) How many possible names are there for an alien from each continent?

c) An alien tells you the first two syllables of their name are "Qo-qo". What is the probability that they are from the Southwest continent, given that the remaining two syllables are equally likely to be any of the 21 syllables in the alien language?

Ric Crossman

[Marks: 100]

LEC005. Tpygorahpcial rerors

When reviewing her lecture notes, Clare spends α minutes fixing the first mistake on each page, and β minutes on every subsequent mistake. Denote by M the number of mistakes on a given page, and by T the total time fixing them.

Write out expressions for $\mathbb{E}[T|M = 0]$, $\mathbb{E}[T|M = 1]$ and $\mathbb{E}[T|M = k]$, where $k \geq 2$, and work out the average time Clare takes to fix one page of lecture notes $\mathbb{E}[T]$, when M is Poisson-ly distributed with parameter λ.

Clare Wallace

[Marks: 100]

LEC006. Battle of Hastings, 14 October 1066

The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof, and woe to the hardy Norman who ventured to enter their redoubts; for a single blow of a Saxon war-hatchet would break his lance and cut through his coat of mail... when Harold threw himself into the fray, the Saxons were one mighty square of men, shouting the battle-cries, 'Ut!', 'Olicrosse!', 'Godemite!'

How big was Harold's army?

Jens Funke

[Marks: 100]

LEC007. The nightmare of the ping pong ball

Following the tragic events of the annual Sphere Packing Competition, Tabitha Table Tennis Ball goes to sleep and dreams that instead of being surrounded by its close friends, Bartek Basketball and Lemon Lemon, it is trapped in a large square room together with its nemesis: evil Fergus Football that wants to trample on it and squash it flat.

Luckily, through the miraculous act of lucid dreaming, Tabitha finds safety in some area of the floor. Given that its diameter is 4 cm, and the diameter of Fergus is 22 cm, will there be any part of the room safe enough for Tabitha to stay in and poke fun at Fergus?

Anna Felikson

[Marks: 100]

LEC008. The larva, my friend, is blowin' in the wind

Eric has grown $n + 1$ sunflowers in his garden, all in a row. Like a good mathematician, he has numbered them with integers 0, 1, 2, etc up to integer n. Eric then places his pet caterpillar, which is very hungry, onto Sunflower 0.

Each day, the wind in Eric's garden blows in such a way that the caterpillar is transferred from its current sunflower to a neighbouring sunflower uniformly at random (ask your lecturer about the fluid mechanics required for this to happen!). In other words, if the caterpillar is currently on Sunflower k (for $0 < k < n$), then on the next day it will be on Sunflower $k + 1$ with probability $1/2$, and on Sunflower $k - 1$ with probability $1/2$. If it is currently on Sunflower 0, then on the next day it will definitely be on Sunflower 1. Assume that the caterpillar changes sunflowers exactly once a day.

What is the expected number of days required for the caterpillar to reach Sunflower n?

Nicholas Georgiou

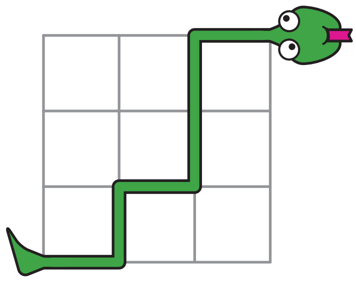
[Marks: 100]

LEC009. Snakes on a plane

Consider a two-dimensional integer grid consisting of the points (i, j) with $i, j \in \{0, 1, 2, \ldots, n\}$. Consider right-up paths on this grid, which are paths from $(0, 0)$ to (n, n) that, at each step, either move (i, j) to $(i + 1, j)$ **or** $(i, j + 1)$. *These would move to the right or up, respectively, if plotted in the usual way, and there are a total number of* ${}^{2n}C_n$ *such possible paths.*

For example, in the grid below, for $n = 3$, one of the ${}^{6}C_3 = 40$ possible right-up paths is

$$(0,0)--(1,0)--(1,1)--(2,1)--$$
$$--(2,2)--(2,3)--(3,3)$$



(a) Derive a (simple) expression for the proportion of right-up paths that, for given $m \in \{0, 1, 2, \ldots, n\}$, go through one or more points (m, j) with $j \geq m$ but do not go through a point (m, j) with $j < m$.

(b) Derive a (simple) expression for the proportion of right-up paths that, for given $m \in \{0, 1, 2, \ldots, n\}$, go through one or more points (m, j) with $j \leq m$ but do not go through a point (m, j) with $j > m$.

Consider now a scenario of trials with binary outcome, 'success' or 'failure', and let (i, j) represent i successes observed in the past and j successes to occur in future, out of n observed trials with interest in n future trials.

(c) Define the function $f_m(j)$, for $m, j \in \{0, 1, 2, \ldots, n\}$, as the proportion of right-up paths that go through (m, j) but not through any (m, k) for $k \neq j$. Is $f_m(j)$, for given m, a probability measure for the number of successes in future, given m successes in the past?

Frank Coolen

[Marks: 100]

LEC010. What do we do with a drunken sailor?

A drunken sailor leaves a pub and walks along a street in one direction. He takes 4 steps before falling asleep, alternating steps with left and right feet. Every time he takes a step, he drags the other foot to where his leading foot is. The sole of his left shoe is dirty with white paint, so both times he takes a step with his right foot the left foot leaves a white trace.

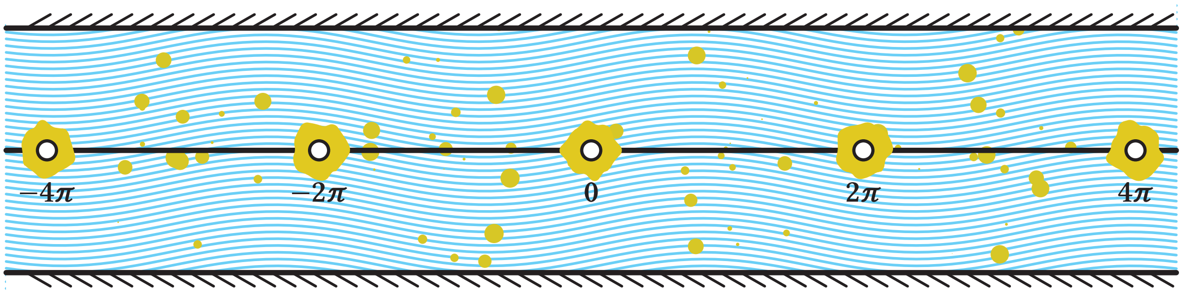
After the sailor fell asleep, it turned out that for any positive real $x < 1$ one can find a segment of length x yards with both endpoints painted in white. What is the minimum possible distance the sailor went with his right foot?

Pavel Tumarkin

[Marks: 100]

LEC011. Pissing in an infinitely long pool

We consider an infinitely long pool of width $2D$ as shown in the schematic figure below:



Infinitely many people with infinitely large bladders are continuously urinating into the pool. As shown in the figure, these people stand at $x = 2n\pi$ and $y = 0$, where n is an integer. An applied mathematician concludes that the urine concentration, denoted by $u(x, y)$, satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -(D-y)^2 \cos x,$$

where u is periodic in the x-direction. The urine cannot penetrate the pool's side walls, which implies

$$\left.\frac{\partial u}{\partial y}\right|_{y=\pm D} = 0.$$

We are given that the urine concentration at the locations where the people stand is U_0, ie $u(2n\pi, 0) = U_0$.

Find $u(x, y)$ in the entire pool.

⟦*Hint*: look for a solution of the form $u(x, y) = A(y)\cos x$.⟧

Hossein Kafiabad

[Marks: 100]

LEC012. Don't go conjugating waterfalls

Let n be a positive even integer. Let $\boldsymbol{J}$ be the matrix with (i, j)-entry equal to 0 unless $j = i + 1$ in which case it is equal to 1. For example, if $n = 4$, then

$$\boldsymbol{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

An $n \times n$ matrix $\boldsymbol{A}$ is called a *waterfall* if there is an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{PAP}^{-1} = \boldsymbol{J}$. A matrix $\boldsymbol{B}$ is called a *barrel* if there is a waterfall $\boldsymbol{A}$ that commutes with it, i.e. such that $\boldsymbol{AB} = \boldsymbol{BA}$.

For $1 \leq i \leq \frac{n}{2}$ let $\boldsymbol{B_i}$ be the matrix obtained from $\boldsymbol{J}$ by replacing the $(i, i+1)$ entry (i.e. the i-th '1') by a zero. You are given that exactly one of these matrices is a barrel. Which one?

Jack Shotton

[Marks: 100]

LEC013.

Prior predictive mass function

For a discrete random variable X, suppose we write the probability mass function of X by

$$\pi(x \mid \theta) = \Pr(X = x)$$

for some continuous parameter θ that has prior density function $\pi(\theta)$. Then the **prior predictive mass function** for X is defined by

$$\pi(x) = \int \pi(x \mid \theta)\pi(\theta)\,\mathrm{d}\theta,$$

and represents our marginal uncertainty about X before we see any data.

Suppose I bought a new pair of gloves this winter. Denote by G the number of gloves left at the end of the season. Consider the model and prior

$$\begin{aligned} G \mid \theta &\sim \mathrm{Bin}(2, \theta), \\ \theta &\sim \mathrm{Beta}(a, a), \end{aligned}$$

where $a > 0$ is a constant to be chosen. At the start of the winter season, I believe that the probability I will have at most one glove left by the end of the winter is 0.7. Using the prior predictive mass function or otherwise, find a value of a that is consistent with this belief.

⟦*Hint*: you may use the fact that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.⟧

Sarah Heaps

[Marks: 100]

LEC014.

You are looking to place a bet on a tennis match between Nolos Djokaraz (N) and Jafael Sindal (J). You have absolutely no idea who will win, who is the favourite, or whether the pair are evenly matched.

There are three types of bets available: B1, B2, and B3, each costing a given amount of money x_1, x_2, x_3. The bets pay out as follows, for some amount q with $0 \leq q \leq 1$:

- B1: If N wins, pays £1, otherwise pays nothing
- B2: If J wins, pays £1, otherwise pays nothing
- B3: If N wins, pays £q, if J wins, pays £$(1-q)$

Your 'friend' and you are going to buy and sell some bets to each other. You can sell or buy any number of bets of each type. You both know the value of q. You get to decide the value of x_1, x_2, x_3. But if you set (say) B1 at price x_1, then (if asked) you must sell any number of bets B1 to your friend at price x_1, or buy any number of bets B1 from your friend.

You set x_1 at price £p_1 and x_2 at price £p_2, with $0 < p_1, p_2 < 1$ and $p_1 + p_2 = 1$. Show there's *only one possible price* for x_3 such that your friend cannot buy or sell bets to you in order to guarantee to make money off you.

For example: if $q = 1/2$ and you priced x_3 at price £1, then your friend can sell you one of each of B1, B2, B3. You pay $x_1 + x_2 + x_3 = p_1 + p_2 + 1 = $ £2 to get one of each bet, but if N wins, you get £1.50, and if J wins, you get £1.50, so you lose money either way.

James Liley

[Marks: 100]

A

Extra resources

What's in a (UK number plate) name?

Most of you will know that the two numbers on a UK number plate will tell you the period of time in which the car was first registered, like the one below that was registered some time between September 2001 and February 2002:

EP51 LON

The table below will reveal that this car would have been issued at the DVLA office in Chelmsford, which you can tell from the **first two letters**. This is called the car's *memory tag*.

Comprehensive list of car registration plate areas

Below is a table displaying all possible memory tags for cars issued between 2001 and 2013, known as the infix system. While the regional DVLA offices are now closed (as we thus have a centralised system), the DVLA still tend to use close-fitting memory tags. Most of the cars registered in Durham, for example, will likely begin with an **N** for **North**.

Memory tag	Region	DVLA office
AA, AB, AC, AD, AE, AF, AG, AH, AJ, AK, AL, AM, AN	Anglia	Peterborough
AO, AP, AR, AS, AT, AU	Anglia	Norwich
AV, AW, AX, AY	Anglia	Ipswich
BA, BB, BC, BD, BE, BF, BG, BH, BJ, BK, BL, BM, BN, BO, BP, BR, BS, BT, BU, BV, BW, BX, BY	Birmingham	Birmingham
CA, CB, CC, CD, CE, CF, CG, CH, CJ, CK, CL, CM, CN, CO	Cymru	Cardiff
CP, CR, CS, CT, CU, CV	Cymru	Swansea
CW, CX, CY	Cymru	Bangor
DA, DB, DC, DD, DE, DF, DG, DH, DJ, DK	Deeside to Shrewsbury	Chester
DL, DM, DN, DO, DP, DR, DS, DT, DU, DV, DW, DX, DY	Deeside to Shrewsbury	Shrewsbury
EA, EB, EC, ED, EE, EF, EG, EH, EJ, EK, EL, EM, EN, EO, EP, ER, ES, ET, EU, EV, EW, EX, EY	Essex	Chelmsford
FA, FB, FC, FD, FE, FF, FG, FH, FJ, FK, FL, FM, FN, FP	Forest and Fens	Nottingham
FR, FS, FT, FV, FW, FX, FY	Forest and Fens	Lincoln
GA, GB, GC, GD, GE, GF, GG, GH, GJ, GK, GL, GM, GN, GO	Gardon of England	Maidstone
GP, GR, GS, GT, GU, GV, GW, GX, GY	Garden of England	Brighton

Memory tag	Region	DVLA office
HA, HB, HC, HD, HE, HF, HG, HH, HJ	Hampshire and Dorset	Bournemouth
HK, HL, HM, HN, HO, HP, HR, HS, HT, HU, HV, HX, HY	Hampshire and Dorset	Portsmouth
HW	Hampshire and Dorset	Isle of Wight
KA, KB, KC, KD, KE, KF, KG, KH, KJ, KK, KL	-	Luton
KM, KN, KO, KP, KR, KS, KT, KU, KV, KW, KX, KY	-	Northampton
LA, LB, LC, LD, LE, LF, LG, LH, LJ	London	Wimbledon
LK, LL, LM, LN, LO, LP, LR, LS, LT	London	Stanmore
LU, LV, LW, LX, LY	London	Sidcup
MA, MB, MC, MD, ME, MF, MG, MH, MJ, MK, ML, MM, MN, MO, MP, MR, MS, MT, MU, MV, MW, MX, MY	Manchester and Merseyside	Manchester
NA, NB, NC, ND, NE, NF, NG, NH, NJ, NK, NL, NM, NN, NO	North	Newcastle
NP, NR, NS, NT, NU, NV, NW, NX, NY	North	Stockton
OA, OB, OC, OD, OE, OF, OG, OH, OJ, OK, OL, OM, ON, OO, OP, OR, OS, OT, OU, OV, OW, OX, OY	Oxford	Oxford
PA, PB, PC, PD, PE, PF, PG, PH, PJ, PK, PL, PM, PN, PO, PP, PR, PS, PT	Preston	Preston
PU, PV, PW, PX, PY	Preston	Carlisle
RA, RB, RC, RD, RE, RF, RG, RH, RJ, RK, RL, RM, RN, RO, RP, RR, RS, RT, RU, RV, RW, RX, RY	Reading	Reading
SA, SB, SC, SD, SE, SF, SG, SH, SJ	Scotland	Glasgow
SK, SL, SM, SN, SO	Scotland	Edinburgh
SP, SR, SS, ST	Scotland	Dundee
SU, SV, SW	Scotland	Aberdeen
SX, SY	Scotland	Inverness
VA, VB, VC, VD, VE, VF, VG, VH, VJ, VK, VL, VM, VN, VO, VP, VR, VS, VT, VU, VV, VW, VX, VY	Severn Valley	Worcester
WA, WB, WC, WD, WE, WF, WG, WH, WJ	West of England	Exeter
WK, WL	West of England	Truro
WM, WN, WO, WP, WR, WS, WT, WU, WV, WW, WX, WY	West of England	Bristol
YA, YB, YC, YD, YE, YF, YG, YH, YJ, YK	Yorkshire	Leeds
YL, YM, YN, YO, YP, YR, YS, YT, YU	Yorkshire	Sheffield
YV, YW, YX, YY	Yorkshire	Barnsley

Oh what a night!

The prefix system was used for cars registered between 1983 and 2001. Their format followed a one letter; one-to-three number; three letter code. The first letter (the *prefix*) signified the year of issue: A for '83, B for '84 etc until Y for '01, excluding the letters I and Q.

The suffix system was used for cars registered between 1963 and 1983. Their format followed a three letter; one-to-three number; one letter code. The final letter (the *suffix*) signified the year of issue: A for '63, B for '64 etc until Y for '83, excluding the letters I and Q.

For both systems, two of the three end letters acted as memory tags.

Normal Distribution Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Here is a table displaying values of the Normal Distribution.

The entry for z is the probability lying to the left of z for a $N(0,1)$ distribution, written as:

$$\begin{aligned}\Phi(z) &= P(Z < z)\\ &= 1 - \Phi(-z).\end{aligned}$$