Spontaneous Symmetry Breaking

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Why break a symmetry?

Mathematical physicists build Lagrangians that encode information about the symmetries of our universe. As an example, a theory for a non-interacting (free) scalar field is symmetric under Lorentz transformations:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

Broken symmetries arise when a symmetry of a field's Lagrangian is not a symmetry of the field's vacuum [1]. In quantum field theory (QFT), where particles exist as quanta of fields, we study these because particle mass terms are not locally symmetric and we therefore cannot add them into our Lagrangians.

This is a problem because some particles are massive; we should hope to to find a way to encode information about particle masses inside Lagrangians. Understanding spontaneous symmetry breaking is a step towards this.

Interacting picture

Consider an interacting theory for a charged scalar field known from QFT:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi - \frac{1}{2}m^2\phi^\dagger\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2.$$

It is clear that change of wavefunction phase $\phi \mapsto e^{i\alpha}\phi$ leaves this Lagrangian invariant. The collection of all possible phase rotations forms the group $U(1)$, leaving us to say this Lagrangian is $U(1)$-invariant.

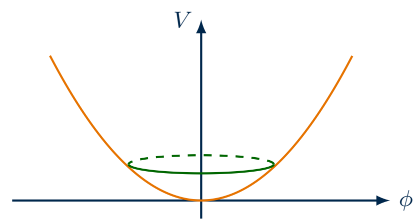
Potential analysis

Take the potential of the aforementioned interacting Lagrangian:

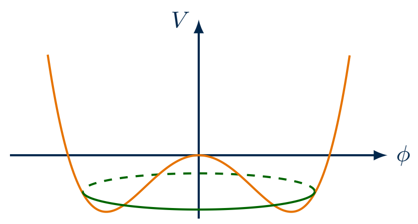
$$V(\phi, \phi^\dagger) = \frac{1}{2}m^2\phi^\dagger\phi + \frac{1}{4}\lambda(\phi^\dagger\phi)^2.$$

In the case of *free theory*, when $\lambda = 0$, we require $m^2 > 0$ for a bounded potential. This draws out the familiar simple harmonic oscillator (SHO) potential. The vacuum sits at the origin, and this single-valued vacuum respects the $U(1)$ symmetry. This differs from the interacting picture. For $\lambda \neq 0$, a bounded potential **demands** $\lambda > 0$ and m^2 now has more freedom.

If we keep m^2 positive then the SHO potential is recovered, albeit steeper. The interesting physics arises when we consider $m^2 < 0$ and uncover a new shaped potential: the Mexican hat, aptly named for its 3D representation.



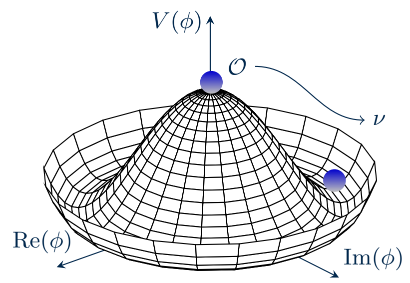
SHO potential.



Mexican hat potential.

The origin is no longer the vacuum/ground state. Instead, there is a ring of minimal values in which non of them are zero. These ground states no longer respect the change of phase symmetry.

What does this tell us?



Spontaneous selection of a ground state ν [2].

What a change of phase symmetry now does is allow us to transform into other ground states, which are just different points in field space. It is in this sense that the symmetry has spontaneously been broken: with no energy, a particle may begin at the origin and be forced into this ring of minimum values [1].

The collection of all ground states forms a **ground state manifold**. In our case, it is a circle:

$$\mathcal{M}_0 := \{\nu : V(\nu) = V_{\text{min}}\} = \mathsf{S}^1.$$

Creating a particle

Now that we understand the structure of the potential, we can discuss the nature of a bare particle that may live in it. A bare particle emerges from exciting an elementary quantum field, such as the one shown above.

First, one can redefine coordinates so that our 'origin' is now a specific ground state. For completely real analysis we should choose $\vartheta = 0$ from the manifold $\mathcal{M}_0$ and call it ν:

$$\nu := \sqrt{-\frac{m^2}{\lambda}}.$$

The potential can then be factored to encode this: $V(\phi) = \frac{1}{4}\lambda(|\phi|^2 - \nu^2)^2\star$.

Because we are working in two-dimensional complex space, the field can be reparametrised via its radial part and angular part. It is key to note that variations in ϑ will take us around $\mathcal{M}_0$ and variations in r will take us up and down the potential walls. With this is mind, the interacting Lagrangian can be rewritten as such:

$$\phi(x) = r(x)e^{i\vartheta(x)} \implies$$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu r(x)\partial^\mu r(x) - \frac{1}{2}r^2\partial_\mu\vartheta\partial^\mu\vartheta - \frac{1}{4}\lambda(r^2 - \nu^2)^2.$$

To create our bare particle, we take our ground state ν and add some radial oscillations: $r(x) = \nu + \eta(x)$. The potential now reads off as the following, with the quadratic part determining the mass of the bare particle η:

$$V(\eta) = \frac{\lambda}{4}\left((\nu+\eta)^2 - \nu^2\right)^2 = \frac{\lambda}{4}\eta^2(\eta + 2\nu)^2.$$

$\star$In practice one can ignore the constant term that falls out from this factoring. We are only concerned with the dynamics provided by the potential, and dynamics involve gradients, eliminating the constants.

Mass spectrum

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}(\nu+\eta)^2\partial_\mu\vartheta\partial^\mu\vartheta - \frac{\lambda}{4}\eta^2(\eta+2\nu)^2.$$

This is the full Lagrangian of the η field. Its mass can be extracted from the coefficient of the quadratic term,

$$m_\eta^2 = 2\lambda\nu^2 = -2m^2 > 0.$$

Notice how the mass of the bare particle differs from what would have been interpreted from the Lagrangian at face value: $m_\eta^2 \neq m^2$ [3]. In some sense the mass of the particle was hidden; it was uncovered via the breaking of a $U(1)$ symmetry.

Goldstone's theorem

By looking at the above Lagrangian it is evident that one of our degrees of freedom $\vartheta(x)$ exists only in gradient form–more specifically, there is no ϑ^2 term.

This has the interpretation of being a massless scalar field. We only encountered this field by breaking the continuous symmetry. There is an underlying theory here. It is a direct result of Goldstone's theorem [4]:

> *Every broken symmetry coming from a continuous group will provide us with a massless boson field, called a* ***Goldstone boson****. The number of Goldstone bosons will be the same as the number of dimensions that makes up the ground state manifold.*

Our ground state manifold $\mathcal{M}_0$ is a circle, a one-dimensional manifold, giving us one massless scalar field: $\vartheta(x)$.

In conclusion

We can conclude that the mass of a particle can only be read off from the Lagrangian if its vacuum state corresponds to the origin. Otherwise, if the vacuum state of the potential is nonzero, one must perform perturbative analysis to find the true masses of the bare particles.

The extra degrees of freedom we have to play with form massless scalar fields, known as Goldstone bosons.

To be continued...

In my report I am analysing spontaneous symmetry breaking in the case of gauged groups and groups in higher dimensions. These are important for understanding how the Standard Model of Particle Physics is built. They give rise to massless fields such as the photon and gluon fields, and also massive ones that govern weak interactions in non-trivial ways.

References

[1] D. Tong. The standard model, 2019. URL `https://www.damtp.cam.ac.uk/user/tong/sm/standardmodel.pdf`.

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[4] A. Salam J. Goldstone and S. Weinberg. Broken symmetries, 1962. URL `https://link.aps.org/doi/10.1103/PhysRev.127.965`.

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